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Entropy Solution to Nonlinear Elliptic Problem with Non-local Boundary Conditions and L^1-data

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Abstract. We study a nonlinear elliptic problem with non-local boundary conditions and L^1-data. We prove an existence and uniqueness result of an entropy solution.

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1 Introduction and assumptions

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N\geq 2$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega=\Gamma_D\cup\Gamma_N$ with $\Gamma_D\cap\Gamma_N=\emptyset$. Our aim is to study the following problem

$$P(\beta,\rho,f,d)\left\{\begin{array}{ll} \beta(u)-\nabla\cdot a(x,\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_D,\\ \left.\begin{array}{l}\rho(u)+\displaystyle\int_{\Gamma_N} a(\cdot,\nabla u)\cdot\eta=d\\ u\equiv \text{cste}\end{array}\right\} & \text{on } \Gamma_N,\end{array}\right.$$

where η is the unit outward normal vector on $\partial\Omega$, β and ρ are two continuous non decreasing functions on $\mathbb{R}$ such that

$$\mathcal{D}(\beta)=\mathcal{D}(\rho)=\mathrm{Im}(\beta)=\mathrm{Im}(\rho)=\mathbb{R}. \tag{1.1}$$

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a is a Leray-Lions operator, f is a function in $L^1(\Omega)$ and $d \in \mathbb{R}$.

Recall that a Leray-Lions type operator is a Carathéodory function $a(x,\xi): \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ (that is $a(x,\xi)$ is continuous with respect to ξ for a.e. $x \in \Omega$ and measurable with respect to x for every $\xi \in \mathbb{R}^N$) and there exists $p \in (1,N)$ such that:

- There exists a positive constant C such that
$$|a(x,\xi)| \leq C(j(x) + |\xi|^{p-1}), \tag{1.2}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'}(\Omega)$, with $1/p + 1/p' = 1$.

- The following inequalities hold
$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \tag{1.3}$$
for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$, and there exists $C' > 0$ such that
$$\frac{1}{C'} |\xi|^p \leq a(x,\xi) \cdot \xi, \tag{1.4}$$
for almost every $x \in \Omega$, and for every $\xi \in \mathbb{R}^N$.

Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or the boundary conditions may contains integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modeling in a 3D stratified petroleum reservoir with arbitrary geometry (see [1] and [2]).

In the main problem considered in this paper, in contrast of the standard case where the condition on the boundary is given on the local values of the flux, nonlocal boundary conditions acts on the average of the flux on the boundary. More precisely, in addition to Dirichlet boundary condition on Γ_D, i.e.
$$u = 0, \qquad \text{on } \Gamma_D, \tag{1.5}$$
u is asking to satisfy the following nonlocal condition
$$\rho(u) + \int_{\Gamma_N} a(x,\nabla u) \cdot \eta = d \qquad \text{on } \Gamma_N. \tag{1.6}$$