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Maximum Principle and Existence of Weak Solutions for Nonlinear System Involving Singular p-Laplacian Operators

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Abstract. In this work we give necessary and sufficient conditions to have a maximum principle for nonlinear system involving singular (p,q)-Laplacian operators on bounded domain Ω of R^n and then we prove the existence of positive weak solutions for the same system by using the Browder theorem method.

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1 Introduction

In this paper, we study the maximum principle and existence of weak solutions for the following nonlinear system involving singular (p,q)-Laplacian operators on bounded domain Ω of R^n

$$\begin{cases} -\operatorname{div}[|x|^{-rp}|\nabla u|^{p-2}\nabla u]=|x|^{-(r+1)p+\gamma}[a|u|^{p-2}u+bh(u,v)]+f, & \text{in } \Omega,\\ -\operatorname{div}[|x|^{-sq}|\nabla v|^{q-2}\nabla v]=|x|^{-(s+1)q+\delta}[ck(u,v)+d|v|^{q-2}v]+g, & \text{in } \Omega,\\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n with boundary $\partial\Omega$, $0\in\Omega$, $1<p,q<n$, $0\le r<\frac{n-p}{p}$, $0\le s<\frac{n-q}{q}$, a,b,c,d,γ,δ are positive constants, $h,k:\mathbb{R}^2\to\mathbb{R}$ are continuous functions which

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have some properties which will be specified later and f,g are given functions. The feature that needs to be highlighted in system (1.1) is the singularity in the weights. This type of problem has received a lot of attention and extensive investigations have been carried out over the years around the questions of existence, nonexistence and uniqueness of solutions. Due to this singularity in the weights, the extensions are challenging and nontrivial. Also, singular nonlinear problems have been of great interest recently. This is because of both the intensive development of the theory of singular calculus itself and the applications of such constructions in various physical fields such as fluid mechanics, glaciology, molecular physics, quantum cosmology and linearization of combustion models (see [1, 2] for example). Such singular nonlinear problems arise naturally and they occupy a central role in the interdisciplinary research between analysis, geometry, biology, elasticity, mathematical physics, etc. A crucial milestone in the understanding of the elliptic problems involving the singular quasilinear elliptic operator $-\mathrm{div}[|\mathrm{x}|^{-\mathrm{rp}}|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}]$ is the paper by Caffarelli, Kohn and Nirenberg [3] (see also [4–6]). For the regular case, that is, when $r=s=0, \gamma=p$ and $\delta=q$; the nonlinear system has been studied by several authors (see [7, 8]).

On the other hand, there have been many papers concerned with the existence of weak solutions for singular elliptic systems in recent years (see [6, 9–12] and related papers in their references.

For the singular problem in unbounded domains, we refer the reader to [13–15], and related papers in their references.

This paper is organized as follows:

In Section 2, we present some hypotheses and technical results which will be used in the sequel. Section 3 is devoted to the study of maximum principle for the system (1.1). Finally, in Section 4, we prove the existence of positive weak solutions for the system (1.1) using the Browder theorem method.

2 Preliminaries and technical results

In this section, we assume the following hypotheses

(H_1) $1<p,q<n$, $0<r<\frac{n-p}{p}$, $0<s<\frac{n-q}{q}$,

(H_2) $\alpha,\beta\geq 0$, $\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1$ and $\gamma,\delta>0$,

(H_3) $a,b,c,d>0$, $b\leq \inf\left(\frac{|x|^{-(s+1)q+\delta}}{|x|^{-(r+1)p+\gamma}}\right)^{\frac{\beta+1}{q}}$ and $c\leq \inf\left(\frac{|x|^{-(r+1)p+\gamma}}{|x|^{-(s+1)q+\delta}}\right)^{\frac{\alpha+1}{p}}$,

(H_4) $f\in L^{p'}(\Omega)$, $g\in L^{q'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$;