JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 3, pp. 208-224

doi: 10.4208/jpde.v28.n3.2
September 2015

An Alternative Method for Solving Lagrange's First-Order Partial Differential Equation with Linear Function Coefficients

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Received 19 March 2015; Accepted 30 July 2015

Abstract. An alternative method of solving Lagrange's first-order partial differential equation of the form

$$(a_1x+b_1y+c_1z)p+(a_2x+b_2y+c_2z)q=a_3x+b_3y+c_3z,$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$ and a_i, b_i, c_i $(i=1,2,3)$ are all real numbers has been presented here.

AMS Subject Classifications: 35A25, 35A99

Chinese Library Classifications: O175.2

Key Words: Lagrange's first-order partial differential equation; linear functions; simultaneous ordinary differential equations; linear homogeneous algebraic equations.

1 Introduction

In the paper [1] by S.M.H. Islam and J. Das, a method of solving the partial differential equation of the form

$$(a_1x+b_1y+c_1z)p+(a_2x+b_2y+c_2z)q=a_3x+b_3y+c_3z, \tag{1.1}$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$ and a_i, b_i, c_i $(i=1,2,3)$ are all real numbers, has been discussed. The present paper comprises a detailed discussion of an alternative method of the same. This method enables us to find the solutions in some cases of failure of the method adopted in the paper [1].

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2 The method

The simultaneous ordinary differential equations corresponding to the PDE (1.1) are

$$\frac{\mathrm{d}x}{a_1x+b_1y+c_1z}=\frac{\mathrm{d}y}{a_2x+b_2y+c_2z}=\frac{\mathrm{d}z}{a_3x+b_3y+c_3z}. \tag{2.1}$$

Suppose that it is possible to find numbers ρ, α_i, β_i, γ_i $(i=1,2,3)\in C$ such that each ratio of (2.1) equals to

$$\begin{aligned}
&\frac{(\alpha_1x+\beta_1y+\gamma_1z)\mathrm{d}x+(\alpha_2x+\beta_2y+\gamma_2z)\mathrm{d}y+(\alpha_3x+\beta_3y+\gamma_3z)\mathrm{d}z}{\begin{array}{l}(\alpha_1x+\beta_1y+\gamma_1z)(a_1x+b_1y+c_1z)+(\alpha_2x+\beta_2y+\gamma_2z)(a_2x+b_2y+c_2z)\\ \quad+(\alpha_3x+\beta_3y+\gamma_3z)(a_3x+b_3y+c_3z)\end{array}}\\
=&\frac{(\alpha_1x+\beta_1y+\gamma_1z)\mathrm{d}x+(\alpha_2x+\beta_2y+\gamma_2z)\mathrm{d}y+(\alpha_3x+\beta_3y+\gamma_3z)\mathrm{d}z}{\begin{array}{l}\left(\sum\limits_{i=1}^{3}\alpha_ia_i\right)x^2+\left(\sum\limits_{i=1}^{3}\beta_ib_i\right)y^2+\left(\sum\limits_{i=1}^{3}\gamma_ic_i\right)z^2+\left(\sum\limits_{i=1}^{3}\alpha_ib_i+\sum\limits_{i=1}^{3}\beta_ia_i\right)xy\\ \quad+\left(\sum\limits_{i=1}^{3}\beta_ic_i+\sum\limits_{i=1}^{3}\gamma_ib_i\right)yz+\left(\sum\limits_{i=1}^{3}\alpha_ic_i+\sum\limits_{i=1}^{3}\gamma_ia_i\right)zx\end{array}}\\
=&\frac{\mathrm{d}D}{\rho D},
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
D=&\left(\sum_{i=1}^{3}\alpha_ia_i\right)x^2+\left(\sum_{i=1}^{3}\beta_ib_i\right)y^2+\left(\sum_{i=1}^{3}\gamma_ic_i\right)z^2+\left(\sum_{i=1}^{3}\alpha_ib_i+\sum_{i=1}^{3}\beta_ia_i\right)xy\\
&+\left(\sum_{i=1}^{3}\beta_ic_i+\sum_{i=1}^{3}\gamma_ib_i\right)yz+\left(\sum_{i=1}^{3}\alpha_ic_i+\sum_{i=1}^{3}\gamma_ia_i\right)zx,
\end{aligned}$$

and $\mathrm{d}D$ denotes the total derivative of D:

$$\begin{aligned}
\mathrm{d}D=&2\left(\sum_{i=1}^{3}\alpha_ia_i\right)x\mathrm{d}x+2\left(\sum_{i=1}^{3}\beta_ib_i\right)y\mathrm{d}y\\
&+2\left(\sum_{i=1}^{3}\gamma_ic_i\right)z\mathrm{d}z+\left(\sum_{i=1}^{3}\alpha_ib_i+\sum_{i=1}^{3}\beta_ia_i\right)(x\mathrm{d}y+y\mathrm{d}x)\\
&+\left(\sum_{i=1}^{3}\beta_ic_i+\sum_{i=1}^{3}\gamma_ib_i\right)(y\mathrm{d}z+z\mathrm{d}y)+\left(\sum_{i=1}^{3}\alpha_ic_i+\sum_{i=1}^{3}\gamma_ia_i\right)(z\mathrm{d}x+x\mathrm{d}z).
\end{aligned}$$