JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **27**, No. 3, pp. 245-250

doi: 10.4208/jpde.v27.n3.5
September 2014

Liquid Crystal Flows with Regularity in One Direction

ZHANG Zujin*

School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China.

Received 7 February 2014; Accepted 9 May 2014

Abstract. In this paper, we consider the Cauchy problem for the model of liquid crystal. We show that if the velocity field $\boldsymbol{u}$ satisfies

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1+\frac{1}{q}, \quad 2<q\leq\infty,$$

then the solution is in fact smooth.

AMS Subject Classifications: 35Q35, 76B03

Chinese Library Classifications: O175.28, O175.29

Key Words: Liquid crystals; regularity criteria.

1 Introduction

In this paper, we consider the following model of liquid crystal in $\mathbb{R}^3$ introduced by Lin [33]:

$$\begin{cases} \boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\triangle\boldsymbol{u}+\nabla\pi=-\nabla\cdot(\nabla\boldsymbol{d}\odot\boldsymbol{d}), & \nabla\cdot\boldsymbol{u}=0, \\ \boldsymbol{d}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}-\triangle\boldsymbol{d}=-\boldsymbol{f}(\boldsymbol{d}), & |\boldsymbol{d}|\leq 1, \\ (\boldsymbol{u},\boldsymbol{d})|_{t=0}=(\boldsymbol{u}_0,\boldsymbol{d}_0), & \nabla\cdot\boldsymbol{u}_0=0,\ |\boldsymbol{d}_0|\leq 1, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the fluid velocity field, $\boldsymbol{d}=(d_1,d_2,d_3)$ is the (averaged) macroscopic/continuum molecule orientation, π is a scalar pressure, $\boldsymbol{u}_0$ and $\boldsymbol{d}_0$ are the prescribed initial data,

$$\boldsymbol{f}(\boldsymbol{d})=\frac{1}{\varepsilon^2}\left(|\boldsymbol{d}|^2-1\right)\boldsymbol{d}, \quad (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_{ij}=\partial_i d_k\partial_j d_k.$$

*Corresponding author. *Email address:* `zhangzujin361@163.com` (Z. Zhang)

Here and thereafter, we use the summation convention that repeated indices are summed automatically over $\{1,2,3\}$.

The existence of a global-in-time weak solution and the local unique strong solution has been established by Lin and Liu [1]. But as for the incompressible Navier-Stokes system ($\boldsymbol{d}$ is constant in (1.1)), whether a given global weak solution is regular and whether the local unique strong solution can exist globally are challenging open problems.

Motivated by the regularity criteria for the Navier-Stokes [2–13] and MHD equations [14–26], some authors considered the regularity conditions for (1.1), see [27–32] and references cited therein.

In this paper, we would like to improve the regularity criterion

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\leq\infty \tag{1.2}$$

established in [29].

Before we state the precise result, let us recall the weak formulation of (1.1).

Definition 1.1 ([1]). *A measurable pair* $(\boldsymbol{u},\boldsymbol{d})$ *is said to be a weak solution of* (1.1) *on* $[0,T]\times\mathbb{R}^3$, *provided the following assertions hold:*

(1) $\boldsymbol{u}\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;

(2) $\boldsymbol{d}\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3))$;

(3) $(1.1)_{1,2}$ *hold in the sense of distributions.*

Now our main result reads:

Theorem 1.1. *Let* $\boldsymbol{u}_0\in L^2(\mathbb{R}^3)$ *satisfy* $\nabla\cdot\boldsymbol{u}_0=0$, $\boldsymbol{d}_0\in H^1(\mathbb{R}^3)$ *with* $|\boldsymbol{d}_0|\leq 1$, *and let the measurable pair* $(\boldsymbol{u},\boldsymbol{d})$ *be a given weak solution of* (1.1) *with initial data* $(\boldsymbol{u}_0,\boldsymbol{d}_0)$. *If*

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1+\frac{1}{q}, \quad 2<q\leq\infty,$$

then the solution is in fact strong, and thus classical.

By a strong solution, we mean $(\boldsymbol{u},\boldsymbol{d})$ satisfy

$$\begin{aligned}\boldsymbol{u}&\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)),\\ \boldsymbol{d}&\in L^\infty(0,T;H^2(\mathbb{R}^3))\cap L^2(0,T;H^3(\mathbb{R}^3)).\end{aligned}$$

Remark 1.1. Notice that

$$\lim_{q\to 2^+}\left(1+\frac{1}{q}\right)=\frac{3}{2},$$

the scaling dimension of Theorem 1.1 can almost achieve $3/2$.