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$W_0^{1,p(x)}$ Versus C^1 Local Minimizers for a Functional with Critical Growth

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Abstract. Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be a bounded smooth domain, p is Hölder continuous on $\overline{\Omega}$,

$$1 < p^- := \inf_{\Omega} p(x) \leq p^+ = \sup_{\Omega} p(x) < \infty,$$

and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(x,s) \geq 0, \forall (x,s) \in \Omega \times \mathbb{R}^+$ and $\sup_{x \in \Omega} f(x,s) \leq C(1+s)^{q(x)}$, $\forall s \in \mathbb{R}^+$, $\forall x \in \Omega$ for some $0 < q(x) \in C(\overline{\Omega})$ satisfying $1 < p(x) < q(x) \leq p^*(x) - 1$, $\forall x \in \overline{\Omega}$ and $1 < p^- \leq p^+ < q^- \leq q^+$. As usual, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$. Consider the functional $I : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined as

$$I(u) \stackrel{\text{def}}{=} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \int_{\Omega} F(x,u^+) \mathrm{d}x, \quad \forall u \in W_0^{1,p(x)}(\Omega),$$

where $F(x,u) = \int_0^s f(x,s) \mathrm{d}s$. Theorem 1.1 proves that if $u_0 \in C^1(\overline{\Omega})$ is a local minimum of I in the $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology, then it is also a local minimum in $W_0^{1,p(x)}(\Omega)$ topology. This result is useful for proving multiple solutions to the associated Euler-lagrange equation (P) defined below.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain, p is Hölder continuous on $\overline{\Omega}$,

$$1 < p_- := \inf_{\Omega} p(x) \leq p_+ = \sup_{\Omega} p(x) < \infty. \tag{1.1}$$

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The assumptions on the source terms f is as follows:

(f1) $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a measurable function with respect to the first argument and continuous differentiable with respect to the second argument for a.e. $x\in\Omega$. Moreover, $f(x,0)=0$ for $(x,s)\in\overline{\Omega}\times\mathbb{R}^+$.

(f2) There exists $q(x)>p(x)-1$ satisfying $q(x)\le p^*(x)-1\overset{\text{def}}{=}\frac{Np(x)}{N-p(x)}-1$ if $p(x)<N$, $q(x)<\infty$ otherwise, and $1<p^-\le p^+<q^-\le q^+$ such that $f(x,s)\le C(1+s)^{q(x)}$ for all $(x,s)\in\Omega\times\mathbb{R}^+$ and for some $C>0$.

Let $F(x,u)\overset{\text{def}}{=}\int_0^u f(x,s)\mathrm{d}s$. We consider functional $I:W_0^{1,p(x)}(\Omega)\to\mathbb{R}$ given by

$$I(u)\overset{\text{def}}{=}\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x-\int_\Omega F(x,u^+)\mathrm{d}x,\quad\forall u\in W_0^{1,p(x)}(\Omega),\tag{1.2}$$

where as usual $t^+\overset{\text{def}}{=}\max(t,0)$.

The operator $\Delta_{p(x)}u:=\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplace where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u:=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where $p>1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. Our aim in this paper is to show the following

Theorem 1.1. *Suppose that $p\in C^{0,\beta}(\overline{\Omega})$ and the conditions* **(f1)-(f2)**, (1.1) *are satisfied. Let $u_0\in C^1(\overline{\Omega})$ be a local minimizer of I in $C^1(\overline{\Omega})\cap C_0(\overline{\Omega})$ topology; that is,*

$$\exists\epsilon>0\text{ such that }u\in C^1(\overline{\Omega})\cap C_0(\overline{\Omega}),\ \|u-u_0\|_{C^1(\overline{\Omega})}<\epsilon\Rightarrow I(u_0)\le I(u).$$

Then, u_0 is a local minimum of I in $W_0^{1,p(x)}(\Omega)$ also.

We remark that u_0 satisfies in the distributions sense the Euler-Lagrange equation associated to I, that is

$$(\mathrm{P})\begin{cases}-\Delta_{p(x)}u=f(x,u), & \text{in }\Omega,\\ u|_{\partial\Omega}=0,\quad u>0, & \text{in }\Omega.\end{cases}$$

It means that $u_0\in W_0^{1,p(x)}(\Omega)$ is a weak solution to (P), i.e. satisfies $\mathrm{essinf}_K u_0>0$ over every compact set $K\subset\Omega$ and

$$\int_\Omega|\nabla u_0|^{p(x)-2}\nabla u_0\cdot\nabla\phi\mathrm{d}x=\int_\Omega f(x,u_0)\phi\mathrm{d}x,\tag{1.3}$$

for all $\phi\in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi:\Omega\to\mathbb{R}$ with compact support. Using the approach introduced in Brezis-Nirenberg [1], used in Ambrosetti-Brezis-Cerami [2] and extended to the p-Laplacian case in Guedda-Veron [3], Azorero-Manfredi-Peral [4], Theorem 1.1 can be used to prove the existence of a second