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On an Anisotropic Equation with Critical Exponent and Non-Standard Growth Condition

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Abstract. Using variational methods, we prove the existence of a nontrivial weak solution for the problem

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\right)=\lambda a(x)|u|^{q(x)-2}u+|u|^{p^*-2}u, & \text{in } \Omega,\\ u=0, & \text{in } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $2\leq p_i<N$, $i=\overline{1,N}$, $q:\overline{\Omega}\to(1,p^*)$ is a continuous function, $p^*=\frac{N}{\sum_{i=1}^{N}\frac{1}{p_i}-1}$ is the critical exponent for this class of problem, and λ is a parameter.

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Key Words: Anisotropic equation; critical exponent; variational methods; existence.

1 Introduction and preliminaries

In this paper, we are interested in the existence of a nontrivial weak solution for the problem

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\right)=\lambda a(x)|u|^{q(x)-2}u+|u|^{p^*-2}u, & \text{in } \Omega,\\ u=0, & \text{in } \partial\Omega,\end{cases} \tag{1.1}$$

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where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $2\leq p_i<N$, $i=\overline{1,N}$, $q:\overline{\Omega}\to(1,p^*)$ is a continuous function,

$$p^*=\frac{N}{\sum_{i=1}^N\frac{1}{p_i}-1},$$

is the critical exponent for this class of problem, and λ is a parameter.

In the case when $p_i(x)=p(x)$ for any $i=1,2,\cdots,N$, the operator involved in (1.1) has similar properties to the $p(x)$-Laplace operator, i.e., $\Delta_{p(x)}u:=\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the isotropic p-Laplace operator $\Delta_p u:=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where $p>1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. The study of nonlinear elliptic problems (equations and systems) involving quasilinear homogeneous type operators like the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{k,p}(\Omega)$ in order to find weak solutions. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. In the case of nonhomogeneous $p(x)$-Laplace operators the natural setting for this approach is the use of the variable exponent Sobolev spaces. Differential and partial differential equations with non-standard growth conditions have received specific attention in recent decades. The interest played by such growth conditions in elastic mechanics and electrorheological fluid dynamics has been highlighted in many physical and mathematical works.

In a recent paper [1], I. Fragalà et al. have studied the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum\limits_{i=1}^N\partial_{x_i}\Big(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\Big)=\lambda u^{p-1}, & \text{in }\Omega,\\ u\geq 0, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega, \end{cases}\tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $p_i>1$ for all $i=1,2,\cdots,N$ and $p>1$. Note that if $p_i=2$ for all $i=1,2,\cdots,N$ then problem (1.2) reduces to the well-known semilinear equation $-\Delta u=\lambda u^{p-1}$. By proving an embedding theorem involving the critical exponent of anisotropic type, the authors obtained some existence and nonexistence results in the case when $p>p_+=\max\{p_1,p_2,\cdots,p_N\}$ or $p<p_-=\min\{p_1,p_2,\cdots,p_N\}$. The results in [1] were extended by A.D. Castro et al. [2], in which the authors studied problem (1.2) in the case when $p_-<p<p_+$. In order to study the existence of solutions for (1.2) the above authors found the solutions in the space $W_0^{1,\overrightarrow{p}}(\Omega)$ which is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i},$$