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Convergence of a Linearized and Conservative Difference Scheme for the Klein-Gordon-Zakharov Equation

WANG Tingchun[1,*] and GUO Boling[2]

[1] *School of Mathematics and Statistics, Nanjing University of Information Science & Technology, Nanjing 210044, Jiangsu, China.*
[2] *Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.*

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Abstract. A linearized and conservative finite difference scheme is presented for the initial-boundary value problem of the Klein-Gordon-Zakharov (KGZ) equation. The new scheme is also decoupled in computation, which means that no iteration is needed and parallel computation can be used, so it is expected to be more efficient in implementation. The existence of the difference solution is proved by Browder fixed point theorem. Besides the standard energy method, in order to overcome the difficulty in obtaining a priori estimate, an induction argument is used to prove that the new scheme is uniquely solvable and second order convergent for U in the discrete L^∞-norm, and for N in the discrete L^2-norm, respectively, where U and N are the numerical solutions of the KGZ equation. Numerical results verify the theoretical analysis.

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1 Introduction

The Klein-Gordon-Zakharov (KGZ) equation

$$\partial_{tt}u-\partial_{xx}u+u+mu+|u|^2u=0, \tag{1.1a}$$

$$\partial_{tt}m-\partial_{xx}m=\partial_{xx}(|u|^2), \tag{1.1b}$$

*Corresponding author. *Email addresses:* `iapcmwang@gmail.com` (T. Wang), `gbl@iapcm.ac.cn` (B. Guo)

is a classical model which describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma [1]. The equations are apparently coupled equations by two functions $u(x,t)$ and $m(x,t)$ to be solved. The function $u(x,t)$ denotes the fast time scale component of electric field raised by electrons and the function $m(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function, and $m(x,t)$ is a real function.

Extensive mathematical and numerical studies have been carried out for the KGZ equation in the literatures. Along the mathematical front, for the well-posedness and global smooth solutions of the KGZ equation, we refer to [2–5] and references therein. Along the numerical front, some conservative finite difference schemes [6–8] have been developed for the KGZ equation. Z. Fei et al. pointed out in [9] that the nonconservative schemes may easily show nonlinear blow-up, and they presented a new conservative linear difference scheme for nonlinear Schrödinger equation. In [10], Li and Vu-Quoc also said, "in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation." In [11–17] the conservative finite difference schemes were used for some nonlinear equations, and the numerical results were very good.

In general, the solutions of (1.1a)-(1.1b) decays rapidly to zero for $|x|\gg 0$ (see [1]). Therefore, numerically we can solve (1.1a)-(1.1b) in a finite domain $\Omega=(x_l,x_r)$ with $-x_l\gg 0$, $x_r\gg 0$. In this paper, we investigate the KGZ equations on $[x_l,x_r]\times[0,T]$ and consider the numerical solution of (1.1a)-(1.1b) subject to initial conditions

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad m(x,0)=m_0(x),\quad m_t(x,0)=m_1(x), \tag{1.2}$$

and boundary conditions

$$u(x_l,t)=u(x_r,t)=0,\quad m(x_l,t)=m(x_r,t)=0, \tag{1.3}$$

where $u_0(x)$, $u_1(x)$, $m_0(x)$ and $m_1(x)$ are known smooth functions. The initial-boundary value problem (1.1a)-(1.3) conserves the *energy*

$$E(t):=\int_{x_l}^{x_r}\Big[|\partial_t u|^2+|\partial_x u|^2+|u|^2+m|u|^2+\frac{1}{2}|v|^2+\frac{1}{2}|m|^2+\frac{1}{2}|u|^4\Big]dx=E(0), \tag{1.4}$$

where v is given by

$$v=-\partial_x w,\quad \partial_{xx}w=\partial_t m. \tag{1.5}$$

For convenience, some notations are firstly introduced. For a positive integer N, choose time-step $\tau=T/N$ and denote time steps $t_n=n\tau$, $n=0,1,2,\cdots,N$; choose mesh size $h=(x_r-x_l)/J$ with J a positive integer and denote grid points as

$$x_j=a+jh,\quad j=0,1,\cdots,J.$$

Denote the index sets

$$\mathcal{T}_J=\{j|j=0,1,2,\cdots,J\},\quad \mathcal{T}_J^o=\{j|j=1,2,\cdots,J-1\}.$$