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Ground State Solutions for a Semilinear Elliptic Equation Involving Concave-Convex Nonlinearities

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Abstract. This work is devoted to the existence and multiplicity properties of the ground state solutions of the semilinear boundary value problem $-\Delta u = \lambda a(x)u|u|^{q-2} + b(x)u|u|^{2^*-2}$ in a bounded domain coupled with Dirichlet boundary condition. Here 2^* is the critical Sobolev exponent, and the term ground state refers to minimizers of the corresponding energy within the set of nontrivial positive solutions. Using the Nehari manifold method we prove that one can find an interval Λ such that there exist at least two positive solutions of the problem for $\lambda \in \Lambda$.

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1 Introduction

We consider the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda a(x)u|u|^{q-2} + b(x)u|u|^{2^*-2}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $\lambda > 0$, $1 \leq q < 2$, and $2^* = 2N/(N-2)$ is the critical Sobolev exponent and the weight functions a,b are satisfying the following conditions:

(A) $a^+ = \max\{a,0\} \not\equiv 0$ and $a \in L^{r_q}(\Omega)$ where $r_q = \frac{r}{r-q}$ for some $r \in (q, 2^*-1)$, with in addition $a(x) \geq 0$ a.e in Ω in case $q=1$;

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(B) $b^+ = \max\{b,0\} \not\equiv 0$ and $b \in C(\overline{\Omega})$.

Tsung-fang Wu [1]has investigated the following equation:

$$\begin{cases} -\Delta u = \lambda a(x) u^q + b(x) u^p, & x \in \Omega, \\ u \geq 0,\ u \not\equiv 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 \leq q < 1 < p < 2^* - 1$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 2$), $\lambda > 0$ and the weight functions a,b satisfy the following conditions:

(A$'$) $a^+ = \max\{a,0\} \not\equiv 0$ and $a \in L^{r_q}(\Omega)$ where $r_q = \frac{r}{r-(q+1)}$ for some $r \in (q+1, 2^*)$, with in addition $a(x) \geq 0$ a.e in Ω in case $q = 0$;

(B$'$) $b^+ = \max\{b,0\} \not\equiv 0$ and $b \in L^{s_p}(\Omega)$ where $s_p = \frac{s}{s-(p+1)}$ for some $s \in (p+1, 2^*)$.

If the weight functions $a \equiv b \equiv 1$, Ambrosetti-Brezis-Cerami [2] studied Eq. (1.2). They established that there exists $\lambda_0 > 0$ such that Eq. (1.2) attains at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Wu [3] found that if the weight functions a changes sign in $\overline{\Omega}$, $b \equiv 1$ and λ is sufficiently small in Eq. (1.2), then Eq. (1.2) has at least two positive solutions.

Throughout this paper we denote $H_0^1(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \| = \| u \|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{1/2}.$$

The function $u \in H_0^1(\Omega)$ is said to be a weak solution of the Eq. (1.1), if u satisfies

$$\int_\Omega \left(\nabla u \nabla v - |u|^{2^*-2} u v - \lambda |u|^{q-2} u v \right) \mathrm{d}x = 0, \quad \forall v \in H_0^1(\Omega).$$

The energy functional corresponding to Eq. (1.1) is defined as follows:

$$J_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{2^*} \int_\Omega b(x) |u|^{2^*} \mathrm{d}x - \frac{\lambda}{q} \int_\Omega a(x) |u|^q \mathrm{d}x,$$

and then J_λ is well defined on $H_0^1(\Omega)$. It is well-known that the solutions of Eq. (1.1) are the critical points of the functional J_λ.

We denote by S_l the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^l(\Omega)$, where $1 \leq l \leq 2^*$. We define the Palais-Smale (or (PS)-) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_λ as follows:

Definition 1.1. (i) *For $c \in \mathbb{R}$, a sequence u_n is a $(PS)_c$-sequence in $H_0^1(\Omega)$ for J_λ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.*

(ii) *$c \in \mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J_λ if there exists a $(PS)_c$-sequence in $H_0^1(\Omega)$ for J_λ.*

(iii) *J_λ satisfies the $(PS)_c$-condition in $H_0^1(\Omega)$ if any $(PS)_c$-sequence u_n in $H_0^1(\Omega)$ for J_λ contains a convergent subsequence.*