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The Freedericksz Transition and the Asymptotic Behavior in Nematic Liquid Crystals

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Abstract. We consider the stability of a specific nematic liquid crystal configuration under an applied magnetic field. We show that for some specific configuration there exist two critical values H_n and H_{sh} of applied magnetic field. When the intensity of the magnetic field is smaller than H_n, the configuration of the energy is only global minimizer, when the intensity is between H_n and H_{sh}, the configuration is not global minimizer, but is weakly stable, and when the intensity is larger than H_{sh}, the configuration is instable. Moreover, we also examine the asymptotic behavior of the global minimizer as the intensity tends to the infinity.

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1 Introduction

The purpose of this paper is to use the Oseen-Frank model to examine the change of stability of a specific nematic liquid crystal configuration under an applied magnetic field. The effect of applied electric and magnetic fields on liquid crystals is an important problem in the physics of liquid crystals. It is well known that as the magnetic field increases passing a critical value the configuration looses its stability. This phenomenon has been studied by many physicists and mathematicians, for example, see Atkin and Stewart [1, 2], Cohen and Luskin [3] and Lin and Pan [4]. The theory for molecular orientation in nematic liquid crystal was given by Ericksen and Leslie [5]. According to the theory, for nematic liquid crystals the bulk free energy without external field is given by

$$\mathcal{W}(\boldsymbol{n}) = \int_\Omega W(\nabla \boldsymbol{n}, \boldsymbol{n}) \mathrm{d}x, \tag{1.1}$$

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where $\boldsymbol{n}=\boldsymbol{n}(x)$ is the unit vector field which called the director at $x\in\Omega$, $\Omega\subset\mathbb{R}^3$ is a bounded smooth domain which is occupied by the material, and $W(\nabla\boldsymbol{n},\boldsymbol{n})$ is the Oseen-Frank energy density:

$$2W(\nabla\boldsymbol{n},\boldsymbol{n})=K_1(\mathrm{div}\boldsymbol{n})^2+K_2(\boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{n})^2+K_3|\boldsymbol{n}\times\mathrm{curl}\boldsymbol{n}|^2+\nu[\mathrm{Tr}(\nabla\boldsymbol{n})^2-(\mathrm{div}\boldsymbol{n})^2], \tag{1.2}$$

where K_i $(i=1,2,3)$ are constants which represent the elastic constants, and ν is a real constant.

Throughout this paper, we impose the strong anchoring condition to the director field, that is to say, the Dirichlet boundary condition $\boldsymbol{n}(x)=\boldsymbol{e}_0(x)$ on the boundary $\partial\Omega$ where $\boldsymbol{e}_0$: $\partial\Omega\to\mathbb{S}^2$ is a given smooth unit vector field. In the situation where liquid crystal material is subject to a static magnetic field $\boldsymbol{H}$, we must add a magnetic energy contribution to the energy $\mathcal{W}(\boldsymbol{n})$. Such a magnetic energy density is of the form $-\chi_a(\boldsymbol{H}\cdot\boldsymbol{n})^2$ where χ_a is a positive constant (cf. de Gennes and Prost [6, p. 287]). We note that under the strong anchoring condition, the integral of the last term of (1.2):

$$\mathcal{S}(\boldsymbol{e}_0):=\int_\Omega[\mathrm{Tr}(\nabla\boldsymbol{u})^2-(\mathrm{div}\boldsymbol{n})^2]\mathrm{d}x, \tag{1.3}$$

represent a surface energy which only depends on the boundary term $\boldsymbol{e}_0$ (cf. Bauman et al. [7]), and so does not affect the problem of finding equilibrium configurations. Thus we consider the total energy density of the material without magnetic field

$$2F(\nabla\boldsymbol{n},\boldsymbol{n})=K_1(\mathrm{div}\boldsymbol{n})^2+K_2(\boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{n})^2+K_3|\boldsymbol{n}\times\mathrm{curl}\boldsymbol{n}|^2.$$

To describe the space of admissible director fields, let $W^{1,2}(\Omega,\mathbb{R}^3)$ be the usual Sobolev space of vector fields, $W^{1,2}(\Omega,\mathbb{S}^2)=\{\boldsymbol{u}\in W^{1,2}(\Omega,\mathbb{R}^3);|\boldsymbol{u}(x)|=1 \text{ a.e. in } \Omega\}$, and

$$W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)=\{\boldsymbol{u}\in W^{1,2}(\Omega,\mathbb{S}^2);\boldsymbol{u}=\boldsymbol{e}_0 \text{ on } \partial\Omega\}.$$

We note that if $\boldsymbol{e}_0:\partial\Omega\to\mathbb{S}^2$ is a smooth vector field and $\partial\Omega$ is Lipschitzian, then $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ is non-empty set (cf. Hardt et al. [8]). By the standard theory of variational problem, we see that the minimizing problem

$$\inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)}\mathcal{F}(\boldsymbol{n}), \tag{1.4a}$$

where

$$\mathcal{F}[\boldsymbol{n}]=\int_\Omega 2F(\nabla\boldsymbol{n},\boldsymbol{n})\mathrm{d}x, \tag{1.4b}$$

is achieved by some $\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$.

We assume that the magnetic field $\boldsymbol{H}$ is of the form $\boldsymbol{H}=\sigma\boldsymbol{h}$ where $\boldsymbol{h}$ is a unit constant vector and $\sigma>0$ is the intensity of $\boldsymbol{H}$ and consider the energy functional

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]&=\mathcal{F}[\boldsymbol{n}]-\chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n})^2\mathrm{d}x\\&=\int_\Omega\{K_1(\mathrm{div}\boldsymbol{n})^2+K_2(\boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{n})^2+K_3|\boldsymbol{n}\times\mathrm{curl}\boldsymbol{n}|^2\}\mathrm{d}x-\chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n})^2\mathrm{d}x.\end{aligned}$$