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Symmetry and Uniqueness of Solutions of an Integral System

ZHANG Zhengce[1,*] and JIANG Minji[2]

[1] *College of Science, Xi'an Jiaotong University, Xi'an 710049, P. R. China.*
[2] *Beijing Areospace Control Center, Beijing 5130-109, P. R. China.*

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Abstract. In this paper, we study the positive solutions for a class of integral systems and prove that all the solutions are radially symmetric and monotonically decreasing about some point. Moreover, we also obtain the uniqueness result for a special case. We use a new type of moving plane method introduced by Chen-Li-Ou [1]. Our new ingredient is the use of Hardy-Littlewood-Sobolev inequality instead of Maximum Principle.

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1 Introduction

In this paper, we study positive solutions of the following system of integral equations in $\mathbb{R}^N$

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^N}\frac{u(y)^k+v(y)^p}{|x-y|^{N-\alpha}}\mathrm{d}y \\ v(x)=\displaystyle\int_{\mathbb{R}^N}\frac{u(y)^q+v(y)^t}{|x-y|^{N-\alpha}}\mathrm{d}y \end{cases} \tag{1.1}$$

with $k=p=q=t=(N+\alpha)/(N-\alpha)$ and $0<\alpha<N$. Under the local integrability conditions $u\in L_{loc}^{2N/(N-\alpha)}(\mathbb{R}^N)$ and $v\in L_{loc}^{2N/(N-\alpha)}(\mathbb{R}^N)$, we first prove that all the solutions are radially symmetric and monotonically decreasing about some point, then we also obtain the uniqueness result for the special case $\alpha=2$. We shall use a new type of moving plane method introduced by Chen-Li-Ou, which technically uses the Hardy-Littlewood-Sobolev inequality instead of Maximum Principle.

*Corresponding author. *Email addresses:* zhangzc@mail.xjtu.edu.cn (Z. Zhang), 2003jmj@163.com (M. Jiang)

The integral system (1.1) is closely related to the system of PDEs

$$\begin{cases} (-\Delta)^{\alpha/2}u=u^k+v^p, \\ (-\Delta)^{\alpha/2}v=u^q+v^t, \end{cases} \qquad u,v>0 \text{ in } \mathbb{R}^N. \tag{1.2}$$

In fact, every positive smooth solution of PDE (1.2) multiplied by a constant satisfies (1.1). This can be easily verified as in the proof of Theorem 4.5 in [1]. We also refer this equivalence to [2] for a system with $\alpha=2$. In fact, in (1.2), we define the positive solution of (1.2) in the distribution sense, i.e., $u,v\in H^{\alpha/2}(\mathbb{R}^N)$ satisfies, for any $\phi\in C_0^\infty$ and $\phi\geq 0$,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}(-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}\phi \mathrm{d}x=\int_{\mathbb{R}^N}[u^k(x)+v^p(x)]\phi(x)\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N}(-\Delta)^{\alpha/4}v(-\Delta)^{\alpha/4}\phi \mathrm{d}x=\int_{\mathbb{R}^N}[u^q(x)+v^t(x)]\phi(x)\mathrm{d}x, \end{cases} \tag{1.3}$$

where

$$\int_{\mathbb{R}^N}(-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}\phi \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^N}(-\Delta)^{\alpha/4}v(-\Delta)^{\alpha/4}\phi \mathrm{d}x$$

are defined by the Fourier transform

$$\int_{\mathbb{R}^N}|\xi|^\alpha \widehat{u}(\xi)\widehat{\phi}(\xi)\mathrm{d}\xi \quad \text{and} \quad \int_{\mathbb{R}^N}|\xi|^\alpha \widehat{u}(\xi)\widehat{\phi}(\xi)\mathrm{d}\xi.$$

Here, $\widehat{u},\widehat{v}$ and $\widehat{\phi}$ are the Fourier transforms of u,v and ϕ, respectively. By taking limits, one can see that (1.3) is also true for any $\phi\in H^{\alpha/2}$.

Since we shall use Hardy-Littlewood-Sobolev inequality to prove radial symmetry and monotonicity of our solutions, we begin by recalling the well-known Hardy-Littlewood-Sobolev inequality. Let λ,s,r be real numbers satisfying $0<\alpha<N$, r, $s>1$, and $\|f\|_p$ be the $L^p(\mathbb{R}^N)$ norm of the function f. We shall write by $\|f\|_{L^p(\Omega)}$ the L^p norm of the function f on the domain Ω. Then the classical Hardy-Littlewood-Sobolev inequality states that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y\leq C\|f\|_r\|g\|_s \tag{1.4}$$

for any $f\in L^r(\mathbb{R}^N)$, $g\in L^s(\mathbb{R}^N)$, and $1/r+1/s=(N+\alpha)/N$. To find the best constant $C=C(\alpha,s,N)$ in the inequality, one can maximize the functional

$$J(f,g)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \tag{1.5}$$

under the constraints $\|f\|_r=\|g\|_s=1$.

There are some related works about this paper. When $k=p=q=t=(N+\alpha)/(N-\alpha)$ and $u(x)=v(x)$, System (1.1) becomes the single equation

$$u(x)=\int_{\mathbb{R}^N}\frac{u(y)^{\frac{N+\alpha}{N-\alpha}}}{|x-y|^{N-\alpha}}\mathrm{d}y, \qquad u>0 \text{ in } \mathbb{R}^N. \tag{1.6}$$