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Local Aronson-Bénilan Estimates for Porous Medium Equations under Ricci Flow

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Abstract. In this work we derive local gradient estimates of the Aronson-Bénilan type for positive solutions of porous medium equations under Ricci flow with bounded Ricci curvature. As an application, we derive a Harnack type inequality.

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1 Introduction

The porous medium equation (PME for short)

$$\partial_t u = \Delta u^m, \tag{1.1}$$

where $m>1$, which cames from physics, plays an important pole in the study of diffusive phenomenon like crowd-avoiding population diffusion, groundwater infiltration, flow of gas in porous media, liquid thin films moving under gravity, heat radiation in plasmas and so on. After mathematicians adding in the study from 1950's, now, we already have rather complete theory for PME about existence, uniqueness, regularity and applications. For more relevant knowledge, we refer the reader to the book [1]. What we want to remind the reader is that most of these studies are in Euclidean space.

Just like other nonlinear problems, the mathematical theory of PME is also based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form [2]

$$\sum_i \frac{\partial}{\partial x^i}\left(mu^{m-2}\frac{\partial u}{\partial x^i}\right) \geq -\frac{\kappa}{t}, \qquad \kappa := \frac{n}{n(m-1)+2} \tag{1.2}$$

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which applies to all positive smooth solutions of (1.1) defined on the whole Euclidean space on the condition that $m>m_c:=1-2/n$. The theory of PME on manifolds is rare. In 2008, Lu, Ni, Vázquez and Villani studied the PME on manifolds [3], they got the following local Aronson-Bénilan estimate:

Theorem 1.1. *Let u be a positive smooth solution to PME* (1.1)*, $m>1$, on the cylinder $Q:=B(\mathcal{O},R)\times[0,T]$. Let v be the pressure and let $v_{\max}^{R,T}:=\max_{B(\mathcal{O},R)\times[0,T]}v$.*

(1) *Assume that Ricci curvature* $\mathrm{Ric}\ge 0$ *on $B(\mathcal{O},R)$. Then, for any $\alpha>1$ we have*

$$\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\le a\alpha^2\left(\frac{1}{t}+\frac{v_{\max}^{R,T}}{R^2}(C_1+C_2(\alpha))\right)$$

on $Q':=B(\mathcal{O},R/2)\times[0,T]$. Here, $a:=n(m-1)/(n(m-1)+2)$, and the positive constants C_1 and $C_2(\alpha)$ depend also on m and n.

(2) *Assume that* $\mathrm{Ric}\ge-(n-1)K^2$ *on $B(\mathcal{O},R)$ for some $K\ge 0$. Then, for any $\alpha>1$, we have that on Q',*

$$\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\le a\alpha^2\left(\frac{1}{t}+C_3(\alpha)K^2v_{\max}^{R,T}\right)+a\alpha^2\frac{v_{\max}^{R,T}}{R^2}\left(C_2(\alpha)+C_1'(KR)\right).$$

Here, a and $C_2(\alpha)$ are as before and the positive constants $C_3(\alpha)$ and $C_1'(KR)$ depend also on m and n.

Acceptable values of the constants are:

$$C_1:=40(m-1)(n+2),\qquad C_2(\alpha):=\frac{200a\alpha^2,m^2}{\alpha-1}\qquad C_3(\alpha):=\frac{(m-1)(n-1)}{\alpha-1},$$
$$C_1'(KR):=40(m-1)[3+(n-1)(1+KR)].$$

Note that $C_1'(0)=C_1$.

In this paper, we will follow closely [3] and derive local gradient estimates for positive bounded solutions of PME on Riemannian manifolds under Ricci flow. The Ricci flow

$$\frac{\partial g_{ij}}{\partial t}=-2R_{ij}$$

was introduced by Hamilton [4] in 1982. In [4–7], Hamilton created and developed the Ricci flow and believed that it could be used to prove the Poincaré conjecture and more general topological classification results in dimension 3, and established the so-called Hamilton's program. The difficulty was to analyze singularities in the Ricci flow. Then, in 2002 and 2003, Perelman [8,9] overcame this difficulty and proved the Poincaré conjecture. In the wonderful work of Perelman, gradient estimates for the fundamental solution of the conjugate heat equation played an important role.