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Very Singular Similarity Solutions and Hermitian Spectral Theory for Semilinear Odd-order PDEs

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Abstract. Asymptotic large- and short-time behavior of solutions of the *linear dispersion equation* $u_t = u_{xxx}$ in $\mathbb{R}\times\mathbb{R}_+$, and its $(2k+1)$th-order extensions are studied. Such a refined scattering is based on a "Hermitian" spectral theory for a pair $\{\mathbf{B},\mathbf{B}^*\}$ of non self-adjoint rescaled operators

$$\mathbf{B}=D_y^3+\frac{1}{3}yD_y+\frac{1}{3}I, \quad \text{and the adjoint one } \mathbf{B}^*=D_y^3-\frac{1}{3}yD_y,$$

with the discrete spectrum $\sigma(\mathbf{B})=\sigma(\mathbf{B}^*)=\{\lambda_l=-l/3, l=0,1,2,\cdots\}$ and eigenfunctions for $\mathbf{B}$, $\{\psi_l(y)=[(-1)^l/\sqrt{l!}]D_y^l\mathrm{Ai}(y), l\geq 0\}$, where $\mathrm{Ai}(y)$ is Airy's classic function. Eigenfunctions of $\mathbf{B}^*$ are then *generalized Hermite polynomials*. Applications to *very singular similarity solutions* (VSSs) of the semilinear dispersion equation with absorption,

$$u_S(x,t)=t^{-\frac{1}{p-1}}f\left(\frac{x}{t^{\frac{1}{3}}}\right): \quad u_t=u_{xxx}-|u|^{p-1}u \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+,\ p>1,$$

and to its higher-order counterparts are presented. The goal is, by using various techniques, to show that there exists a countable sequence of critical exponents $\{p_l=1+3/(l+1), l=0,1,2,\cdots\}$ such that, at each $p=p_l$, a p-branch of VSSs bifurcates from the corresponding eigenfunction ψ_l of the linear operator $\mathbf{B}$ above.

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1 Introduction: semilinear odd-order models, history, and results

1.1 Basic dispersion models and applications

As a first basic model, we will study higher odd-order partial differential equations (PDEs) of the form

$$u_t=(-1)^{k+1}D_x^{2k+1}u+\tilde{g}(u) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+, \quad k=1,2,\cdots, \tag{1.1}$$

with bounded integrable initial data $u(x,0)=u_0(x)$ in $\mathbb{R}$. Here $D_x^m=(\partial/\partial x)^m$ denotes the m^{th} partial derivative in x. The odd-order semilinear dispersion equation (1.1) can be considered as a counterpart of the better known semilinear higher-even-order *parabolic* PDE of *reaction-diffusion* type,

$$u_t=(-1)^{k+1}D_x^{2k}u+\tilde{g}(u) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+, \quad k=1,2,\cdots. \tag{1.2}$$

For $k=1$, (1.2) becomes a standard reaction-diffusion equation from combustion theory

$$u_t=u_{xx}+\tilde{g}(u) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+, \tag{1.3}$$

to which dozens of well-known monographs are devoted to. These parabolic equations indeed belong to an entirely different type of PDEs and were much better studied in the twentieth century. However, the analogy between odd and even-order PDEs, such as (1.1) and (1.2), is rather fruitful and will be used later on.

The function (a nonlinear operator) $\tilde{g}(u)$ in (1.2) usually corresponds to absorption-reaction type phenomena and sometimes is assumed to include differential terms, such as $D_x^m u$, with $m<2k+1$ (although we do not consider such cases). It is worth mentioning again that, besides some special and completely integrable PDEs, general odd-order models such as (1.1) are less studied in the mathematical literature, than the parabolic even-order ones (1.2).

Indeed, the most classical example of such an odd-order equation is the *KdV equation*:

$$u_t=u_{xxx}+uu_x \quad (\tilde{g}(u):=uu_x), \tag{1.4}$$

which was introduced by Boussinesq in 1872 together with its soliton solution [1]. The KdV equation models long waves in shallow water and generates a hierarchy of other more complicated PDEs with linear and nonlinear dispersion (dispersive) mechanisms. See further amazing historical aspects concerning (1.4) and related integrable PDEs in [2, p. 226-229].

Concerning higher-order extensions, these naturally appear in classic theory of integrable PDEs from shallow water applications, including the *fifth-order KdV equation*,

$$u_t+u_{xxxxx}+30u^2u_x+20u_xu_{xx}+10uu_{xxx}=0.$$