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Positive Solutions for Singular Quasilinear Schrödinger Equations with One Parameter, II

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Abstract. We establish the existence of positive bound state solutions for the singular quasilinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\mathrm{div}\Big(\rho(|\nabla\psi|^2)\nabla\psi\Big)+\varpi(|\psi|^2)\psi-\lambda\rho(|\psi|^2)\psi,\qquad x\in\Omega,\ t>0,$$

where $\varpi(\tau^2)\tau\to+\infty$ as $\tau\to 0$ and, $\lambda>0$ is a parameter and Ω is a ball in $\mathbb{R}^N$. This problem is studied in connection with the following quasilinear eigenvalue problem with Dirichlet boundary condition

$$-\mathrm{div}\Big(\rho(|\nabla\Psi|^2)\nabla\Psi\Big)=\lambda_1\rho(|\Psi|^2)\Psi,\qquad x\in\Omega.$$

Indeed, we establish the existence of solutions for the above Schrödinger equation when λ belongs to a certain neighborhood of the first eigenvalue λ_1 of this eigenvalue problem. The main feature of this paper is that the nonlinearity $\varpi(|\psi|^2)\psi$ is unbounded around the origin and also the presence of the second order nonlinear term. Our analysis shows the importance of the role played by the parameter λ combined with the nonlinear nonhomogeneous term $\mathrm{div}(\rho(|\nabla\psi|^2)\nabla\psi)$ which leads us to treat this problem in an appropriate Orlicz space. The proofs are based on various techniques related to variational methods and implicit function theorem.

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1 Introduction

This paper is a continuation of [1] where the second named author and J. M. do Ó established the existence of positive bound state solutions for the following class of quasilinear Schrödinger equations

$$i\frac{\partial\psi}{\partial t}=-\Delta\psi+\psi+\varpi(|\psi|^2)\psi-\kappa\Delta\zeta(|\psi|^2)\zeta'(|\psi|^2)\psi, \tag{1.1}$$

where $\psi=\psi(t,x)$, $\psi:\mathbb{R}\times\Omega\longrightarrow\mathbf{C}$, $\kappa>0$ is a constant and Ω is a ball in $\mathbb{R}^N$, $(N\geq 2)$. In this paper we consider the case

$$i\frac{\partial\psi}{\partial t}=-\operatorname{div}\Big(\rho(|\nabla\psi|^2)\nabla\psi\Big)+\varpi(|\psi|^2)\psi-\lambda\rho(|\psi|^2)\psi, \tag{1.2}$$

where ρ is possibly nonhomogeneous.

Quasilinear equations of the form (1.1) and (1.2) arise in several areas of physics in correspondence to different type of functions ρ and ζ: the classical semilinear Schrödinger equations is of the form (1.2) with $\rho(s)=1$, the superfluid-film equation in plasma physics exhibits this structure for $\zeta(s)=s$ while for $\zeta(s)=(1+s)^{1/2}$ equation (1.1) models the self-channeling of a high-power ultra short laser pulse in a plasma [2, 3]. For further physical motivations and developing of the physical aspects we refer to [4–7] and references therein.

Here we consider the case where $s\rho(s^2)$ is an odd nondecreasing function and $\varpi(|\psi|^2)\cdot\psi=g(s)$ where g has singularity at $s=0$. Looking for standing wave solutions of (1.2), from which one gets a solitary traveling wave by exploiting the Lorentz invariance of equation, we set $\psi(t,x)=e^{-i\xi t}u(x)$, where $\xi\in\mathbb{R}$ and $u>0$ is a real function. With this ansatz, one obtains a corresponding equation of elliptic type which has the formal variational structure

$$-\operatorname{div}\Big(\rho(|\nabla u|^2)\nabla u\Big)+g(u)-\lambda\rho(|u|^2)u=0, \qquad u>0, \quad x\in\Omega, \tag{1.3}$$

with Dirichlet boundary condition. Problem (1.3) is studied in connection with the corresponding nonlinear eigenvalue problem

$$-\operatorname{div}\Big(\rho(|\nabla\Psi|^2)\nabla\Psi\Big)=\lambda_1\rho(|\Psi|^2)\Psi, \qquad x\in\Omega, \tag{1.4}$$

Note first that as a consequence of the results in [8], under appropriate assumptions on ρ, Eq. (1.4) posses a ground state solution Ψ which is strictly positive, spherically symmetric and radially decreasing. Indeed, we establish the existence of a positive solution of (1.3) when λ belongs to a certain neighborhood of the first eigenvalue λ_1 of (1.4).

Motivated by the afore mentioned physical aspects, equation (1.1) and (1.2) has recently attracted a lot of attention. Indeed, there are many recent works regarding the existence, qualitative behavior and multiplicity of solutions at which these equations have been studied. We refer the interested reader to [9] for existence results using constraint