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Light-Like Extremal Surfaces in Minkowski Space $\mathbb{R}^{1+(1+n)}$

HE Yijie*

School of Mathematics Sciences, Fudan University, Shanghai 200433, China.

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Abstract. In this paper we investigate the equations for light-like extremal surfaces in Minkowski space $\mathbb{R}^{1+(1+n)}$. We show that the light-like assumption is compatible with the Cauchy problem and give a necessary and sufficient condition on the global existence of classical solutions of the Cauchy problem. Based on this, we obtain entire light-like extremal surfaces by solving the Cauchy problem explicitly when such necessary and sufficient condition holds. Finally, some discussions and related remarks are given.

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Key Words: Light-like; extremal surface; Cauchy problem; global classical solution.

1 Introduction and main results

The investigation of extremal surfaces in the Minkowski space is of great interest in mathematics and physics. Recently it attracted much attention mainly because of its extensive relations to electrodynamics, classical hydrodynamics, general relativity and the string theory (cf. [1–4]).

In mathematics, the extremal surfaces in Minkowski space $\mathbb{R}^{1+(1+n)}$ include the following four types: space-like, time-like, light-like and mixed type. For the time-like case, the surface can be defined as the critical point of the area functional

$$I_n=\iint\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}\mathrm{d}x\mathrm{d}t. \tag{1.1}$$

where

$$\phi=(\phi_1,\phi_2,\cdots,\phi_n)^T,\quad \text{and}\quad \phi_t\cdot\phi_x=\sum_{i=1}^n\phi_{i,t}\phi_{i,x}.$$

*Corresponding author. *Email address:* 072018034@fudan.edu.cn (Y. He)

The corresponding Euler-Lagrange equation gives

$$\begin{aligned}&\left(\frac{(1+|\phi_x|^2)\phi_t-(\phi_t\cdot\phi_x)\phi_x}{\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\right)_t\\&\quad-\left(\frac{(1-|\phi_t|^2)\phi_x+(\phi_t\cdot\phi_x)\phi_t}{\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\right)_x=0,\end{aligned}\tag{1.2}$$

which can be reduced as

$$(1+|\phi_x|^2)\phi_{tt}-2(\phi_t\cdot\phi_x)\phi_{tx}-(1-|\phi_t|^2)\phi_{xx}=0.\tag{1.3}$$

Eq. (1.3) also describes the time-like surfaces with vanishing mean curvature. Based on the geometric properties of this system, Kong et al. [3] presented a necessary and sufficient condition on the global existence of classical solutions of the Cauchy problem.

As for the light-like case, the Euler-Lagrange equation (1.2) is not well-defined because the area density

$$\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}$$

is always zero; meanwhile the induced metrics of light-like surfaces are degenerate by definition such that it is not clear how to define the mean curvature and what kind of light-like surfaces may be treated as extremal. Fortunately, the system (1.3) is still valid to describe a kind of light-like surfaces which can be viewed as natural analogues of time-like extremal surfaces.

Up to now, the time-like case, the space-like case and the case of mixed type have been investigated extensively, see, e.g., Barbashov et al. [5], Milnor [6] and Kong et al. [3] for the time-like case, Calabi [7] and Cheng and Yau [8] for the space-like case and Gu [9,10] for the case of mixed type. In addition, for the multidimensional cases or more general framework, we refer to the papers by Hoppe [11], Lindblad [12] and Chae and Huh [13].

However, only a few results are known for the light-like case. Recently Kong et al. gave only a brief discussion on the light-like extremal surfaces in [4]. Huang and Kong discussed the three-dimensional light-like extremal submanifolds in the Minkowski space $\mathbb{R}^{1+n}(n\geq 3)$ in [14] and obtained the only two classes of light-like extremal submanifolds under a relativistic background. Gorkavyy [15] discussed another way to define minimal light-like surfaces in Minkowski space by applying one particular deformability property of surfaces.

In this paper, we study the Cauchy problem for the system (1.3) with the given initial data

$$\begin{cases}\phi(0,x)=p(x),\\ \phi_t(0,x)=q(x),\end{cases}\tag{1.4}$$

where p and q are smooth vector-valued functions satisfying the light-like condition

$$(1+|p'|^2)(1-|q|^2)+(p'\cdot q)^2\equiv 0.\tag{1.5}$$