Regularity of Radial Solutions to the Complex Hessian Equations

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1 Introduction

Let Ω be a bounded domain in \( \mathbb{C}^n \), and let \( u \in C^2(\Omega) \) be a real valued-function. Then the complex Hessian of \( u \) defined by

\[
[u_{ij}] = \left[ \frac{\partial^2 u(z)}{\partial z_i \partial \bar{z}_j} \right]
\]

is an \( n \times n \) Hermitian matrix at each point \( z \in \Omega \). Let \( H_k \) denote the complex Hessian operator in \( \mathbb{C}^n \), which is defined for \( C^2 \) functions \( u \) as follows:

\[
H_k[u] = \sigma_k(u_{ij}), \quad 1 \leq k \leq n,
\]

where \( \sigma_k \) is the \( k \)-th elementary symmetric function for the eigenvalues of Hessian matrix \( [u_{ij}] \). That is, for \( 1 \leq k \leq n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \),

\[
\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},
\]

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which coincides with the Laplace \( H_1[u] = \Delta u \) if \( k = 1 \), and the Monge-Ampere operator \( H_n[u] = \det(u_{ij}) \) if \( k = n \). We also define \( \sigma_0 = 1, \sigma_k = 0, \forall k > n \) (see, e.g., [3]).

**Definition 1.1.** Let \( A \) be an \( n \times n \) real symmetric matrix, and denote a symmetric convex cone as 
\[
\Gamma_k = \{ A : \sigma_j(A) > 0, \ 1 \leq j \leq k \}.
\]
Then we say \( u \) is \( k \)-subharmonic if the complex Hessian \( H[u] \in \bar{\Gamma}_k \). We also say that \( u \) is plurisubharmonic if \( k = n \) and subharmonic if \( k = 1 \).

We introduce some properties about \( \sigma_k \) for later proof (also see, e.g., [3]).

**Property 1.** Denote \( \sigma_k(\lambda | i) \) as taking \( \lambda_i = 0 \) in \( \sigma_k(\lambda) \). For \( 1 \leq k, i \leq n \) and \( \lambda \in \mathbb{R}^n \)
\[
\sigma_k(\lambda) = \sigma_k(\lambda | i) + \lambda_i \sigma_{k-1}(\lambda | i).
\]

**Property 2.** For all \( \lambda \in \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \ 1 \leq j \leq k \} \), with \( 2 \leq k \leq n \), we have
\[
\sigma_{l-1}(\lambda) \geq \sigma_{l}(\lambda) \sigma_{l-2}(\lambda), \quad \forall \ 2 \leq l \leq k.
\]

We consider the following Dirichlet problem for \( 2 \leq k \leq n \) :
\[
\begin{cases}
u \text{ is } k \text{-subharmonic},
\quad H_k[u] = f, \quad x \in \Omega, \\
u = \phi, \quad x \in \partial \Omega,
\end{cases}
\tag{1.1}
\]
where \( f \in C^n(\bar{\Omega}) \) is non-negative, \( \phi \in C^\infty(\partial \Omega) \), and \( \Omega \) is \( \Gamma_k \)-pseudoconvex with smooth boundary (\( k = n \) i.e. strongly pseudoconvex, see, [4]). The condition \( u \) be \( k \)-subharmonic is imposed for uniqueness (see, [4, 5]). When \( k = n \), the corresponding equation is complex Monge-Ampere equation which has been studied by many authors (see, e.g., [6–10]). One of important results is given by Caffarelli et al. [11] which proves that there exists a \( C^\infty \) solution to this problem provided that \( f \in C^\infty \) is non-vanishing on \( \bar{\Omega} \). The result has recently been generalized by Li [4] to the \( k \)-Hessian operator (in fact more cases). However, when \( f \) is degenerate this is not always true. In this paper we consider what happens in the special case where \( f \geq 0 \) is radially symmetric.

The problem can be stated as follows. Let \( B \) denote the unit ball in \( C^n \). Given \( f \geq 0 \) on \( B \), find a \( k \)-subharmonic function \( u \in C^2(B) \) such that
\[
\begin{cases}
H_k[u] = f(|z|), \quad z \in B, \\
u = 0, \quad z \in \partial B.
\end{cases}
\tag{1.2}
\]

A radial function \( u \) can be considered simply as a function of one real variable \( r \). So in Section 2, we will compute \( H_k[u] \) directly, obtaining a non-linear ordinary differential equation \( H_k[u](r) = f(r) \). This equation is then solved by two integrations, giving \( u \) in terms of \( f \). We have following results