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Regional, Single Point, and Global Blow-Up for the Fourth-Order Porous Medium Type Equation with Source

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Abstract. Blow-up behaviour for the fourth-order quasilinear porous medium equation with source,

$$u_t = -(|u|^n u)_{xxxx} + |u|^{p-1} u \quad \text{in } \mathbb{R}\times\mathbb{R}_+,$$

where $n>0$, $p>1$, is studied. Countable and finite families of similarity blow-up patterns of the form

$$u_S(x,t) = (T-t)^{-\frac{1}{p-1}} f(y), \quad \text{where} \quad y = \frac{x}{(T-t)^{\beta}}, \ \beta = \frac{p-(n+1)}{4(p-1)},$$

which blow-up as $t\to T^- < \infty$, are described. These solutions explain key features of regional (for $p=n+1$), single point (for $p>n+1$), and global (for $p\in(1,n+1)$) blow-up. The concepts and various variational, bifurcation, and numerical approaches for revealing the structure and multiplicities of such blow-up patterns are presented.

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1 Introduction: blow-up reaction-diffusion models

1.1 On classic second-order blow-up models and higher-order diffusion

Blow-up phenomena as intermediate asymptotics and approximations of highly non-stationary processes are common and well known in various areas of mechanics and

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physics. The origin of intensive systematic studies of such nonlinear effects was gas dynamics (since the end of the 1930s and 1940s) supported later in the 1960s by plasma physics (wave collapse) and nonlinear optics (self-focusing phenomena). Nevertheless, it was *reaction-diffusion theory* (going back in itself to fundamental results by Frank-Kamenetskii and first blow-up ODE conclusions by Todes in the 1930s; see below) that exerted the strongest influence on mathematical blow-up research since the 1960s and 70s. It is not an exaggeration to say that reaction-diffusion theory proposed basic and nowadays canonical models, which eventually led to qualitative and rigorous description of principles of formation of blow-up and other singularities in nonlinear PDEs.

Finite-time blow-up singularities lie in the heart of several principal problems of PDE theory concerning existence, uniqueness, optimal regularity, and free-boundary propagation. The role of blow-up analysis in nonlinear PDE theory will increase when more complicated classes of higher-order degenerate parabolic, hyperbolic, nonlinear dispersion, and other equations of interest are involved in the framework of massive mathematical research and application. For such general classes of equations with typically non-potential, non-divergent, and non-monotone operators (see classic monographs by Berger [1] and Krasnosel'skii-Zabreiko [2] for fundamentals of nonlinear operator theory), applications of many known classic techniques associated with some remarkable and famous specific PDEs become non-applicable, so that a principally new methodology and even philosophy of nonlinear PDEs via blow-up analysis should play a part reasonably soon; nobody can postpone this to happen, otherwise this will lead to a retarding PDE theory, which will not be able to answer principle questions to come.

REACTION-DIFFUSION MODELS WITH BLOW-UP.

The second-order quasilinear reaction-diffusion equations from combustion theory are widely used in the mathematical literature and many applications. This class of parabolic PDEs of principal interest in the twentieth century includes classic models beginning with the *Frank-Kamenetskii equation* with exponential nonlinearity (the so-called *solid fuel model*, 1938 [3]),

$$u_t = \triangle u + e^u, \tag{1.1a}$$

$$u_t = \triangle u + u^p, \tag{1.1b}$$

$$u_t = \triangle (u^{n+1}) + u^p, \tag{1.1c}$$

$$u_t = \nabla \cdot (|\nabla u|^n \nabla u) + u^p \tag{1.1d}$$

etc., where $n > 0$ and $p > 1$ are fixed exponents, and similar equations with more general nonlinearities. Due to the superlinear behaviour of the source terms e^u or u^p ($p > 1$ often) as $u \to +\infty$, these PDEs are known to create finite-time blow-up in the sense that a bounded solution ceases to exist and

$$\sup_{x \in \mathbb{R}} u(x,t) \to +\infty \quad \text{as} \quad t \to T^- < \infty. \tag{1.2}$$