Asymptotically Self-Similar Global Solutions for a Higher-Order Semilinear Parabolic System

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Abstract. In this paper, we study the higher-order semilinear parabolic system

\[
\begin{align*}
&u_t + (-\Delta)^m u = a|v|^{p-1}v, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\
v_t + (-\Delta)^m v = b|u|^{q-1}u, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\
u(0,x) = \varphi(x), \quad v(0,x) = \psi(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]

where \(m, p, q > 1, a, b \in \mathbb{R}\). We prove that the global existence of mild solutions for small initial data with respect to certain norms. Some of these solutions are proved to be asymptotically self-similar.

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Key Words: Higher-order parabolic equation; mild global solutions; asymptotically self-similar.

1 Introduction

This article is concerned with the Cauchy problem for the higher-order semilinear parabolic system

\[
\begin{align*}
&u_t + (-\Delta)^m u = a|v|^{p-1}v, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\
v_t + (-\Delta)^m v = b|u|^{q-1}u, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\
u(0,x) = \varphi(x), \quad v(0,x) = \psi(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]

where \(m, p, q > 1, a, b \in \mathbb{R}\). Higher-order semilinear and quasilinear heat equations appear in numerous applications such as thin film theory, flame propagation, bi-stable phase

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transition and higher-order diffusion. We refer reader to the monograph [1] for some of these mathematical models. For studies of higher-order heat equations we refer also to [2–8] and the references therein.

Consider the following nonlinear parabolic equation:

\[
\begin{cases}
    u_t + (-\triangle)^m u = |u|^p, & (t, x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\
    u(0, x) = \varphi(x), & x \in \mathbb{R}^N.
\end{cases}
\]

It is well known that \( p_F = \frac{1 + 2m}{N} \) is the critical Fujita exponent, see [2]. This model was later generalized to a weakly coupled system (1.1), where \( a|v|^{p-1}v \) and \( b|u|^{q-1}u \) are replaced by \( |v|^p \) and \( |u|^q \), respectively, see [4, 5]. In [4], it was shown that for \( \frac{N}{2m} > \max \left\{ \frac{1 + p}{pq - 1}, \frac{1 + q}{pq - 1} \right\} \), there is a global solution with small initial data; while for \( \frac{N}{2m} \leq \max \left\{ \frac{1 + p}{pq - 1}, \frac{1 + q}{pq - 1} \right\} \) and the positive energy on initial data, the solution will blow up in finite time. With the above condition, the life span of solution was studied in [5]. The technique used in [2] and [4] was based on a strongly continuous semigroup \( S(t) \) generated by the infinitesimal generator \( -(-\triangle)^m \). With it and the standard semigroup theory, problem (1.1) can be written as the following equivalent integral system:

\[
\begin{align*}
    u(t) &= S(t)\varphi + a \int_0^t S(t-\tau)(|v(\tau)|^{p-1}v(\tau))d\tau, \\
    v(t) &= S(t)\psi + b \int_0^t S(t-\tau)(|u(\tau)|^{q-1}u(\tau))d\tau,
\end{align*}
\]

(1.2)

where

\[
S(t)\varphi = b(t, \cdot) * \varphi, \quad b(t, x) = t^{-N/(2m)} f(y), \quad y = x/t^{1/(2m)}.
\]

(1.3)

In this paper, we shall prove the global existence of solutions for (1.1) with initial data \( \Phi = (\varphi, \psi) \) small with respect to norm \( \mathcal{N} \) (see (3.3)). In particular, if \( \varphi \) and \( \psi \) are homogeneous of degree \( -2m(1 + p)/(pq - 1) \) and \( -2m(1 + q)/(pq - 1) \) respectively, then the resulting solution of (1.1) is self-similar. In addition, we prove that the initial data of the form \( (1 - \eta)\varphi \) and \( (1 - \eta)\psi \), where \( \eta \) is a cut-off function, and \( \varphi \) and \( \psi \) are homogeneous of degree \( -2m(1 + p)/(pq - 1) \) and \( -2m(1 + q)/(pq - 1) \) respectively, give rise to asymptotically self-similar solutions. Our approach to self-similar solutions of (1.1) is based on the integral system (1.2) and contraction mapping. This is different from that exploited in previous works such as [6–8] on self-similar solutions, which are based on an analysis of the ordinary differential equation verified by the profile of the self-similar solution. The