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SINGULAR SOLUTION OF A QUASILINEAR CONVECTION DIFFUSION DEGENERATE PARABOLIC EQUATION WITH ABSORPTION

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Abstract In this paper the existence and nonexistence of non-trivial solution for the Cauchy problem of the form

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \dfrac{\partial}{\partial x_i} b_i(u) - u^q, & (x,t) \in S_T = R^N \times (0,T), \\ u(x,0) = 0, & x \in R^N \setminus \{0\} \end{cases}$$

are studied. We assume that $|b_i'(s)| \leq M s^{m-1}$, and proved that if $p > 2$, $0 < q < p - 1 + \frac{p}{N}$, $0 \leq m < p - 1 + \frac{p}{N}$, then the problem has a solution; if $p > 2$, $q > p - 1 + \frac{p}{N}$, $0 \leq m \leq \frac{q(p+Np-N-1)}{p+Np-N}$, then the problem has no solution; if $p > 2$,$p - 1 < q < p - 1 + \frac{p}{N}$, $0 \leq m < q$, then the problem has a very singular solution; if $p > 2$, $q > p - 1 + \frac{p}{N}$, $0 < m < q - \frac{p}{2N}$, then the problem has no very singular solution. We use P.D.E. methods such as regularization, Moser iteration and Imbedding Theorem.
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1. Introduction

In this paper we consider the Cauchy problem

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \frac{\partial}{\partial x_i} b_i(u) - u^q \quad \text{in } S_T = R^N \times (0,T), \tag{1.1}$$

$$u(x,0) = 0 \quad x \in R^N \setminus \{0\} \tag{1.2}$$

where $p > 2$, $q > 0$, and $b_i(s) \in C^1(R)$.

The equation (1.1) is a prototype of a certain class of degenerate equations and appears to be relevant to the theory of non-Newtonian fluids. When the initial datum is a measure ,the case is also a model for physical phenomena. The goal of this paper is to give necessary and sufficient conditions which guarantee that (1.1)(1.2) has a non-trivial solution. For the case when $p = 2$, $b_i(u) = 0$, it was shown in [1] that if $0 < q < 1 + \frac{2}{N}$, the problem (1.1)(1.2) has a solution satisfying the initial condition

$$u(x, 0) = \delta(x) \tag{1.3}$$

where $\delta(x)$ denotes the Dirac mass centered at the origin, and that if $q \geq 1 + \frac{2}{N}$, the problem (1.1)(1.3) has no solution. In addition, it was shown in [2] and [3] that if $p \geq 2$ and $p-1 < q < p-1+\frac{p}{N}$, the equation (1.1) has a very singular solution, i.e.,a solution ω with the following properties:

$$\omega \in C(\bar{S}_T \backslash \{(0,0)\}) \tag{1.4}$$

$$\omega(x, 0) = 0 \quad \text{ if } \ x \in R^N \backslash \{0\} \tag{1.5}$$

$$\lim_{t \to 0^+} \int_{|x|<r} \omega(x, t)\mathrm{d}x = \infty \quad \ \forall\ r > 0. \tag{1.6}$$

In this paper, we are interested in the effect of the convection term $\frac{\partial b_i}{\partial x_i}(u)$ on the existence and nonexistence of singular solution to Cauchy problem (1.1)(1.2). For the case $p = 2$, $b_i(u) = u^m$, [4][5] proved that if $1 < q < 1 + \frac{2}{N}$ and $1 < m < 1 + \frac{1}{N}$, (1.1)(1.3) has a unique solution and that if $q \geq 1 + \frac{2}{N}$ and $1 < m \leq \frac{q+1}{2}$,(1.1)(1.2) has no singular solution and if $1 < q < 1 + \frac{2}{N}$, $1 < m \leq \frac{q+1}{2}$, then (1.1)(1.2) has a very singular solution. Here we use the method similar to that in [6].We assume that

$$|b_i^{'}(s)| \leq M s^{m-1} \quad \text{ if } \ s \geq 0. \tag{1.7}$$

We shall prove the following theorems:

Theorem 1 *Suppose that* (1.7) *holds and let* $p > 2$, $0 < q < p - 1 + \frac{p}{N}$, $0 \leq m < p - 1 + \frac{p}{N}$, *then* (1.1)(1.3) *has a solution;*

Theorem 2 *Suppose that* (1.7) *holds and let* $p > 2$, $q > p - 1 + \frac{p}{N}$, $0 \leq m \leq \frac{q(p+Np-N-1)}{p+Np-N}$, *then* (1.1)(1.3) *has no solution;*

Theorem 3 *Suppose that* (1.7) *holds and let* $p > 2$, $p - 1 < q < p - 1 + \frac{p}{N}$, $0 \leq m < q$, *then* (1.1)(1.2) *has a very singular solution;*

Theorem 4 *Suppose that* (1.7) *holds and let* $p > 2$, $q > p-1+\frac{p}{N}$, $0 < m < q-\frac{p}{2N}$, *then* (1.1)(1.2) *has no very singular solution.*

2. Proof of Theorem 1

Definition 2.1 *A solution u of* (1.1)(1.3) *is a nonnegative function defined in S_T such that:*

1. $u \in C(0, T; L^1(R^N)) \bigcap L^\infty(R^N \times (\tau, T)) \bigcap C(\bar{S}_T \backslash \{(0, 0)\})$, $u \in L^p_{loc}(0, T; W^{1,p}(R^N))$, $u_t \in L^1(R^N \times (\tau, T))\ \forall \tau > 0$;