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POLAR COORDINATES FOR THE GENERALIZED BAOUENDI-GRUSHIN OPERATOR AND APPLICATIONS*

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Abstract In this parer, by using the polar coordinates for the generalized Baouendi-Grushin operator

$$\mathcal{L}_\alpha = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{m} |x|^{2\alpha} \frac{\partial^2}{\partial y_j^2},$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, $\alpha > 0$, we obtain the volume of the ball associated to $\mathcal{L}_\alpha$ and prove the nonexistence for a second order evolution inequality which is relative to $\mathcal{L}_\alpha$.

Key Words Generalized Baouendi-Grushin operator; polar coordinate; nonexistence; second order evolution inequality.

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1. Introduction

The polar coordinates for the Heisenberg group $\mathbb{H}_1$ and for the Heisenberg group $\mathbb{H}_n$ were defined by Greiner [1] and D'Ambrosio [2], respectively. In [3] and [4], such coordinates for the Grushin operator in $\mathbb{R}^{n+1}$ and generalized Baouendi-Grushin operator

$$\mathcal{L}_\alpha = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{m} |x|^{2\alpha} \frac{\partial^2}{\partial y_j^2} \tag{1.1}$$

were studied, where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, $\alpha > 0$.

Nonexistence results of positive solutions for singular elliptic inequality, parabolic and hyperbolic inequality in the Euclidean space $\mathbb{R}^n$ have been largely considered, see

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[5, 6] and their references. The singular sub-Laplace inequality and related evolution inequalities in the Heisenberg group $\mathbb{H}_n$ were studied in [2, 7]. The singular sub-elliptic inequality and the first order evolution inequality related to $\mathcal{L}_\alpha$ were considered in [4].

In this parer we will give some applications of the polar coordinates for the operator $\mathcal{L}_\alpha$. In particular, we explicitly compute the volume of the ball in the sense of the distance associated with $\mathcal{L}_\alpha$. Also, we discuss the nonexistence for a second order evolution inequality which is relative to $\mathcal{L}_\alpha$

$$\begin{cases} u_{tt} - \frac{d^2}{\psi_{2\alpha}}\mathcal{L}_\alpha(au) \geqslant |u|^q, & \text{on } \mathbb{R}_*^{n+m} \times (0, +\infty), \\ u(x,y,0) = u_0(x,y), & \text{on } \mathbb{R}_*^{n+m}, \\ u_t(x,y,0) = u_1(x,y), & \text{on } \mathbb{R}_*^{n+m}, \end{cases} \tag{1.2}$$

where $a \in L^\infty(\mathbb{R}^{n+m} \times [0, +\infty))$, $q \geqslant 1$, $\mathbb{R}_*^{n+m} = \mathbb{R}^{n+m} \backslash \{(0,0)\}$. Our consideration is motivated by D'Ambrosio [2]. Let us note that some essential differences appearing unavoidably. We recall some known facts about the operator $\mathcal{L}_\alpha$ (see[8]). Let

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^\alpha \frac{\partial}{\partial y_j} \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{1.3}$$

Denote the generalized gradient

$$\nabla_L = (Z_1, \ldots, Z_n, Z_{n+1} \ldots, Z_{n+m}).$$

There exists a natural family of anisotropic dilations attached to $\mathcal{L}_\alpha$, i.e.,

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y), \qquad \lambda > 0, (x, y) \in \mathbb{R}^{n+m}.$$

It leads to a homogeneous dimension for $\mathcal{L}_\alpha$

$$Q = n + (\alpha + 1)m.$$

One easily examines that

$$\mathcal{L}_\alpha \circ \delta_\lambda = \lambda^2 \delta_\lambda \circ \mathcal{L}_\alpha,$$

so that $\mathcal{L}_\alpha$ becomes homogeneous of degree two with respect to the anisotropic dilations.

Introduce the distance function

$$d(x, y) = (|x|^{2(\alpha+1)} + (\alpha + 1)^2 |y|^2)^{\frac{1}{2(\alpha+1)}}. \tag{1.4}$$

It should also be noted that

$$|\nabla_L d|^2 = \psi_{2\alpha} = \frac{|x|^{2\alpha}}{d^{2\alpha}} \tag{1.5}$$

and

$$\mathcal{L}_\alpha d = \psi_{2\alpha} \frac{Q - 1}{d}. \tag{1.6}$$