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GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS TO THE CAUCHY PROBLEM ON A SEMI-BOUNDED INITIAL AXIS FOR INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract In this paper, we consider the Cauchy problem with initial data given on a semi-bounded axis for inhomogeneous quasilinear hyperbolic systems. Under the assumption that the rightmost (resp. leftmost) eigenvalue is weakly linearly degenerate and the inhomogeneous term satisfies the corresponding matching condition, we obtain the global existence and uniqueness of C^1 solution with small and decaying initial data.
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1. Introduction and Main Result

Consider the following first order inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=F(u), \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$ is the unknown vector function of (t,x), $A(u)$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j=1,\cdots,n)$, $F(u)=(F_1(u),\cdots,F_n(u))^T$ is a given vector function of u with suitably smooth elements $F_i(u)$ and

$$F(0)=0,\quad \nabla F(0)=0. \tag{1.2}$$

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u),\cdots,\lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i=1,\cdots,n$, let $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))$ (resp. $r_i(u)=(r_{i1}(u),\cdots,r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u) \tag{1.3}$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \tag{1.4}$$

we have

$$\det|l_{ij}(u)| \neq 0 \qquad (\text{resp.} \quad \det|r_{ij}(u)| \neq 0). \tag{1.5}$$

Without loss of generality, we assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

We suppose that all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

In particular, if, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.7}$$

the system (1.1) is called to be strictly hyperbolic.

For the Cauchy problem of the system (1.1) with the initial data

$$t = 0: \quad u = \phi(x) \quad (x \geq 0), \tag{1.8}$$

where $\phi(x)$ is a C^1 vector function with bounded C^1 norm, it is proved in [1] that if in a neighbourhood of $u = 0$, $F(u) \equiv 0$ and

$$\lambda_1(u), \cdots, \lambda_{n-1}(u) < \lambda_n(u), \tag{1.9}$$

then for any given initial data satisfying the following decaying property:

$$\theta \triangleq \sup_{x \geq 0}\{(1 + x)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < \infty, \tag{1.10}$$

where $\mu > 0$ is a constant, there exists $\theta_0 > 0$ so small that for any $\theta \in [0, \theta_0]$, Cauchy problem (1.1) and (1.8) admits a unique global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $D = \{(t, x)|\ t \geq 0, x \geq x_n(t)\}$, where $x = x_n(t)$ is the n-th characteristic passing through the origin $O(0, 0)$:

$$\begin{cases} \dfrac{\mathrm{d}x_n(t)}{\mathrm{d}t} = \lambda_n(u(t, x_n(t))) \\ x_n(0) = 0 \end{cases} \tag{1.11}$$

if and only if $\lambda_n(u)$ is weakly linearly degenerate (WLD). On the other hand, if $F(u) \not\equiv 0$, it is proved in [2], [3] that for a strictly hyperbolic system, under the assumptions that the system is weakly linearly degenerate and $F(u) \in C^2$ satisfies the matching condition, the Cauchy problem (1.1) with initial data $\phi(x)(x \in R)$ satisfying

$$\theta \triangleq \sup_{x \in R}\{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} << 1$$