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THE DISSIPATIVE QUASI-GEOSTROPHIC EQUATION IN SPACES ADMITTING SINGULAR SOLUTIONS*

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Abstract This paper studies the Cauchy problem of the dissipative quasi-geostrophic equation in pseudomeasure space $PM^{n+1-2\alpha}(\mathbb{R}^n)$ or Lorentz space $L^{\frac{n}{2\alpha-1},\infty}(\mathbb{R}^n)$, which admit the singular solutions. The global well-posedness is established provided initial data $\theta_0(x)$ are small enough in these spaces. Moreover, the asymptotic stability of solutions in pseudomeasure space is proved. In particular, if the initial data are homogeneous functions of degree $1-2\alpha$, the self-similar solutions are also obtained.

Key Words Dissipative quasi-geostrophic equation; singular solutions; pseudomeasure spaces; Lorentz space; global well-posedness.

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1. Introduction

In this paper we discuss the following Cauchy problem of the dissipative quasi-geostrophic equation:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta + \kappa(-\triangle)^{\alpha}\theta = 0, & (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ \theta(x,0)=\theta_0(x), \end{cases} \tag{1.1}$$

where $0<\alpha\le 1$ is a fixed parameter and $\kappa>0$ is the dissipative coefficient. The function $\theta(t,x)$ represents the potential temperature and the fluid velocity $u(t,x)=$

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$(u_1,\ u_2,\ \cdots,\ u_n)$ is divergence free and determined from $\theta(t,x)$ by a stream function $\psi(t,x)$:

$$u_j=\pm\frac{\partial\psi}{\partial x_k},\ j,\ k=1,\ 2,\ \cdots,\ n. \tag{1.2}$$

The function $\psi(t,x)$ satisfies

$$(-\triangle)^{\frac{1}{2}}\psi=-\theta. \tag{1.3}$$

From (1.2) and (1.3) we can obtain that

$$u_j=\pm\mathcal{R}_{\pi(j)}\theta,\ \pi(j)\text{ is a permutation of } j=1,\ 2,\ \cdots,\ n, \tag{1.4}$$

where u_j may take either plus or minus sign, and

$$\mathcal{R}_j=\frac{-\partial_{x_j}}{(-\triangle)^{1/2}},\ j=1,\ 2,\ \cdots,\ n \tag{1.5}$$

are the Riesz transforms.

In the equation (1.1) the Riesz potential operator $(-\triangle)^{\alpha}$ is defined through the Fourier transform

$$\widehat{(-\triangle)^{\alpha}f}(\xi)=|\xi|^{2\alpha}\hat{f}(\xi), \tag{1.6}$$

where $\hat{f}(\xi)\triangleq\mathcal{F}(f)(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\mathrm{e}^{-ix\xi}f(x)\mathrm{d}x$ is the Fourier transform of $f(x)$. For notational convenience, we occasionally write Λ for $(-\triangle)^{\frac{1}{2}}$. In the particular case $n=2$, we have

$$u(t,x)=\big(\partial_{x_2}(-\triangle)^{-\frac{1}{2}}\theta,-\partial_{x_1}(-\triangle)^{-\frac{1}{2}}\theta\big)=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta)=\mathcal{R}^{\perp}\theta, \tag{1.7}$$

where $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2)$ is the 2-D Riesz transform.

Physically, the 2-D quasi-geostrophic equation models the evolution of temperature of atmospheric and oceanic fluid flow on the two dimensional boundary of a fast rotating three dimensional half space with small Rossby and Ekman number [1, 2]. The scalar $\theta(t,x)$ represents the potential temperature and $u(t,x)$ is the fluid velocity. These equations have been actively investigated because of the mathematical importance and potential applications to meteorology and oceanography [1–4].

Mathematically, the 2-D quasi-geostrophic equation serves as a lower dimensional models of the 3-D Navier-Stokes equations because of the striking similarity between the behavior of its solutions and that of the potential singular solutions of the 3-D N-S equations [1], thus the study of the 2-D quasi-geostrophic equation may provide some clues to the millennium prize problems on the Navier-Stokes equations [5].

In a series of previous works of Wu [6–9], the well-posedness results for initial data θ_0 in Lebesgue space L^p, homogeneous Sobolev space $\dot{L}_{s,p}$, Morrey space $M_{p,\lambda}$ and Hölder