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RICCI FLOW ON SURFACES WITH DEGENERATE INITIAL METRICS

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Abstract It is proved that given a conformal metric $e^{u_0}g_0$, with $e^{u_0} \in L^\infty$, on a 2-dim closed Riemannian manfold (M, g_0), there exists a unique smooth solution $u(t)$ of the Ricci flow such that $u(t) \to u_0$ in L^2 as $t \to 0$.

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In Kähler geometry, one of the central problems is the existence problem of the Calabi's extremal Kähler metric ([1],[2]). The well-known Kähler-Einstein metrics are special extremal metrics for which the existence problem can be reduced to a second order complex Monge-Ampere equation. For the general extremal metric, the equation is either 4th or 6th order depending on whether the scalar curvature function is constant. However, there is a variational structure associated to the extremal metric, which, according to E. Calabi[2], is local minimizer of the so called "Calabi enegy" (i.e., the L^2 norm of the scalar curvature). More recently, S. K. Donaldson[3] and the first named author[4] proved that the extremal Kähler metric is the global minimizer of the Calabi energy respectively. A similar result holds with the Calabi energy replaced by the K-energy ([5]), see[3] and [4].

To attack the existence problem, we may pick a minimizing sequence of Kähler metrics for the K energy or the Calabi energy. Suppose that the Kähler potentials of this sequence converges in weak $C^{1,1}$ topology to a limit $C^{1,1}$ Kähler potential. (In general we do not know if such convergence should be true.) Then, we face the problem that the limit Kähler metric might be degenerated. It is a conjecture of the first named

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author that any $C^{1,1}$ minimizer of the K-energy functional must be smooth. The proof of the conjecture seems hopeless with traditional approaches. The conjecture can be proved if the Kähler Ricci flow can be initiated for any $C^{1,1}$ Kähler potentials and is smooth for positive time.

For general closed Kähler manifolds with positive first Chern class, Chen-Tian [6] is able to construct a weak Kähler-Ricci flow initiated from any L^∞ Kähler metric with $C^{1,1}$ potential. They proved that

1. the Kähler potential under the flow remains uniformly bounded in $C^{1,1}$;

2. the volume form of the weak flow converges strongly in L^2 to the volume of the initial Kähler metric as $t \to 0$.

It is natural to ask if the weak Kähler Ricci flow will become smooth immediately after $t > 0$? This question is answered positively in this note in the special case of Riemannian surface. In a forthcoming paper [7], the first named author and G. Tian prove this for all dimensions along with other results. The significance of such results is that, the original Kähler-Ricci flow defined in the open positive cone of Kähler potentials can actually be extended to the boundary of the cone; and the flow initiated at the boundary will leave the boundary right away and enter into the interior of the cone. Thus, the extended (weak) flow not only preserves the cone structure but also has a wonderful effect of regularizing a weak Kähler metric.

The proof in this note is quite different from that in [7], especially the "regularization" part. We believe the techniques used in our proof here may be useful in other nonlinear geometric problems, such as the yamabe flow, the σ_k problems in conformal geometry and the Minkonski problems for convex hypersurfaces in Euclidean spaces.

For convenience of our presentation, we concentreate on the case of S^2. However, the result holds for any closed surfaces. The main theorem in this paper is:

Theorem 1 *Let g_0 be a smooth Riemannian metric on S^2 and $e^{u_0} \in L^\infty(S^2)$ such that*

$$\int_{S^2} e^{u_0} = \int_{S^2} \mathrm{d}A_0.$$

where dA_0 denotes the area element of g_0. Then the Ricci flow with initial metric $g_0' = e^{u_0} g_0$ has a unique solution $g(t) = e^{u(t)} g_0$ such that $u(t)$ is smooth for $t > 0$ and $e^{u(t)} \to e^{u_0}$ in L^2 as $t \to 0$.

Recall that if $g = e^u g_0$ (with smooth function u) and R_0 is the Gaussian curvature of g_0, then the Gaussian curvature R of g given by

$$K = e^{-u}(K_0 - \frac{1}{2}\Delta u),$$