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THE FREE BOUNDARY PROBLEM IN BIOLOGICAL PHENOMENA*

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Abstract We prove the local existence of weak solution of a free boundary problem for a hyperbolic-parabolic system .

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1. Introduction

In this paper, we consider a free boundary problem arising from some biological phenomena. In absence of any external signal, the spread of a population density $u(x,t)$ is described by the diffusion equation

$$u_t = d\Delta u \tag{1}$$

where $d > 0$ is the diffusion constant. We define the net flux as $j = -d\nabla u$. If there is some external signal S, we just assume that it results in a chemotactic velocity β, then the flux is

$$j = -d\nabla u + \beta u. \tag{2}$$

To be more specific, we assume that the chemotactic velocity β has the direction of the gradient ∇S and that the sensitivity χ to the gradient depends on the signal concentration $S(x,t)$, then $\beta = \chi(S)\nabla S$ (see[1,2]).

Then we obtain the parabolic chemotaxis equation

$$u_t = \nabla(d\nabla u - \chi(S)\nabla S \cdot u). \tag{3}$$

We assume that the spatial spread of the external signal is driven by wave, then the full system for u and S reads as

$$u_t = \nabla(d\nabla u - \chi(S)\nabla S \cdot u) \tag{4}$$

$$\tau S_{tt} = \alpha\Delta S + g(S,u). \tag{5}$$

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The time constant $0 \le \tau \le 1$ indicates that the spatial spread of the organisms u and the signal S are on different time scales. The case $\tau = 0$ corresponds to a quasi-steady state assumption for the signal distribution.

Let $\Omega \subset R^n$ be a bounded open domain and $\Omega_0 \subset\subset \Omega$ be an open domain. Assume a population density $u(x,0)$ to occupies the domain Ω_0 , out of Ω_0 the population density $u(x,0) \equiv 0$ and the external signal S to occupies Ω . For $t > 0$, $u(x,t)$ spreads to domain $\Omega_t \subset \Omega$. Let $\partial\Omega_t$ denote the boundary of Ω_t and n_t denote the outer normal vector of $\partial\Omega_t$, then $\Gamma = \partial\Omega_t \times (0,T)$ is the free boundary.

Suppose that the net flux of population density u is ku on $\partial\Omega_t$, namely

$$-d\nabla u \cdot n_t = ku \quad \partial\Omega_t. \tag{6}$$

On the other hand, notice that the full flux on $\partial\Omega_t$ is

$$j = -d\nabla u \cdot n_t + \chi(S)u\nabla S \cdot n_t. \tag{7}$$

By conservation of population, one has

$$uv_{n_t} = -d\nabla u \cdot n_t + \chi(S)u\nabla S \cdot n_t \quad \text{on} \quad \partial\Omega_t \tag{8}$$

where v_{n_t} is the normal diffusion velocity of $\partial\Omega_t$.

Assume $\Gamma : \Phi(x,t) = 0$, then

$$\begin{aligned} v_{n_t} &= (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt}) \cdot n_t \\ &= (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt}) \cdot \frac{\nabla\Phi}{|\nabla\Phi|} \end{aligned} \tag{9}$$

where $x = (x_1, x_2, \cdots, x_n)$ and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$.

Notice that

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial\Phi}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{\partial\Phi}{\partial x_n} \cdot \frac{dx_n}{dt} = 0. \tag{10}$$

Thus (9) and (10) indicate

$$v_{n_t} = -\frac{1}{|\nabla\Phi|} \cdot \frac{\partial\Phi}{\partial t}. \tag{11}$$

We use (11) in (8) to obtain

$$u\frac{\partial\Phi}{\partial t} = d\nabla u \cdot \nabla\Phi - \chi(S)u\nabla S \cdot \nabla\Phi \quad \text{on} \quad \partial\Omega_t. \tag{12}$$

At last we obtain the conditions of the free boundary Γ

$$-d\nabla u \cdot \frac{\nabla\Phi}{|\nabla\Phi|} = ku \quad \text{on} \quad \Gamma \tag{13}$$