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BV SOLUTIONS TO A DEGENERATE EQUATION COUPLED WITH TIME-DELAY REGULARIZATION *

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Abstract In this paper, we study a initial value problem of a degenerate parabolic equation coupled with time-delay regularization. The existence of BV solutions to the problem for an initial BV_{loc} data is obtained. Moreover, the existence and uniqueness of spatial-periodic classical solutions to its corresponding regularized problem are also given.

Key Words Time-delay; parabolic regularization; BV solution.
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1. Introduction

Image restoration is one of the most important tasks in image processing and computer vision. In many applications, a given image,which is denoted by a function $u_0 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, Ω is typically a rectangular, can be written as a sum $u(x) + \eta(x), x \in \Omega$, where u denotes important components of the given data formed by homogeneous regions with sharp boundaries, while η represents the noise or some components of u_0 which is less interesting. We assume in our paper that η stands for a white additive Gaussian noise. The problem is then to extract the u component from u_0.

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PDE-based regularization and geometry-driven flows have effectively applied to compute the optimal piecewise smooth approximation from the noisy data. One approach to obtain PDE for image restoration starts with setting axioms extracted from some fundamental laws in physics based on desired image properties. In this category, we mention Witkin, Koenderink's scale space theory [1-3] and Perona-Malik's anisotropic diffusion model [4] for examples. In [5], Alvarez, Guichard, Lions and Morel further establish the connection between scale space analysis and PDEs, and prove that the restored image should be the viscosity solution to a mean curvature flow.

By setting the best energy according to ones need, another possibility is the gradient descent flow of the Euler-Lagrange equation(sometimes modified) associated with its minimization problem. In this category, we mention Rudin-Osher-Fatemi [6], Chan-Strong [7], Vese-Osher [8], Sochen-Kimmel-Malladi [9], among many others. In [9-11], Sochen, Kimmel, malladi, Yezzi and El-Fallah, Ford introduce the concept of images as embedded maps and minimal surfaces to the field. In order to obtain edge preserving and noise removing, Yezzi, Sochen, Kimmel and Malladi and El-Fallah, Ford propose the following mean curvature flow

$$\frac{\partial u}{\partial t} = r(x)^{\gamma}\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

with an initial data $u(x,0)=u_0(x)$, where $r(x)=\frac{1}{\sqrt{1+|Du|^2}}$, $\gamma=-1,0,\text{or } 1$, D denotes the gradient operator.

Based on the idea of Vogel, Oman [12], Chan, Strong [7], and Bose, Chen [13], we shall consider the following TV-penalized least square minimization problem

$$\min_u E_{\beta,\lambda}(u) =: \frac{\lambda}{2}\int_\Omega |u-u_0|^2dx + \int_\Omega \alpha(x)\sqrt{|\nabla u|^2+\beta^2}dx. \tag{1}$$

The first term in $E_{\beta,\lambda}(u)$ measures the fidelity to the initial data, the second is a smoothing term with spatial adaptivity. $\beta>0$, $\lambda>0$ are constants. We can see that when $\alpha\equiv 1$, $E_{1,0}$ is non other than the area of the image surface. The function $\alpha(x)$ is chosen to be inversely proportional to the possibility of the presence of edge at the point $x\in\Omega$. Typically, we should choose the the function $\alpha(x)$ in the following form

$$\varphi(\overrightarrow{\nu}) = \frac{1}{1+\frac{|\overrightarrow{\nu}|^2}{K}},$$

where the vector-valued function $\overrightarrow{\nu}$, which approximates the image gradient Du, satisfies the system

$$\overrightarrow{\nu}_t = \frac{1}{\tau}(DG*u-\overrightarrow{\nu}). \tag{2}$$

$K>0$ is a threshold parameter and $\overrightarrow{\nu}$ plays the role of time-delay regularization determined by the positive parameter τ. $G(x)$ is chosen as a smooth function with