J. Partial Diff. Eqs. **19**(2006), 200–207
 Vol.19 No.3

NOTE ON GROUND STATES OF NONLINEAR SCHRÖDINGER SYSTEMS

Thomas Bartsch
(Mathematisches Institut, Universität Giessen, 35392 Giessen, Germany)
(E-mail: Thomas.Bartsch@math.uni-giessen.de)
Zhi-Qiang Wang
(Department of Mathematics and Statistics, Utah State University, Logan, UT 84322, USA)
(E-mail: zhi-qiang.wang@usu.edu)

Dedicated to Professor K. C. Chang on his seventieth birthday

(Received Apr. 25, 2006)

Abstract We give sufficient and necessary conditions for the existence and nonexistence of positive ground state solutions of a class of coupled nonlinear Schrödinger equations.

Key Words Ground states; Schrödinger systems.
2000 MR Subject Classification 35B40, 35B45, 35J40, 92C40.
Chinese Library Classification O175.29

In this note, we are concerned with standing wave solutions $(\Phi_1, \dots, \Phi_N) : \mathbb{R}^n \to \mathbb{C}^N$ of the time-dependent system of N coupled nonlinear Schrödinger equations given by

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j + \displaystyle\sum_{i=1}^{N} \beta_{ij}|\Phi_i|^2\Phi_j & \text{for } \ y \in \mathbb{R}^n, \ t > 0, \\ \Phi_j(y,t) \to 0 & \text{as } \ |y| \to +\infty, \ t > 0, \end{cases} \tag{1.1}$$

$j = 1, \dots, N$, where β_{ij} are nonnegative constants satisfying $\beta_{ij} = \beta_{ji}$, $n \le 3$, $N \ge 2$.

A solitary wave of this system is a solution with $\Phi_j(y,t) = e^{i\lambda_j t}u_j(y)$, $j = 1, \dots, N$. This ansatz leads to the elliptic system

$$\Delta u_j - \lambda_j u_j + \sum_{i=1}^{N} \beta_{ij}u_i^2 u_j = 0, \ \text{in } \mathbb{R}^n, \ j = 1, \dots, N. \tag{1.2}$$

Throughout the paper, λ_j are positive constants. We are interested in solutions $\vec{u} = (u_1, \dots, u_N)$ of (1.2) having all components $u_j > 0$. These are called *positive* solutions as opposed to *semi-positive* solutions which satisfy $u_j \ge 0$ for all j and $u_j > 0$ for at least one j. Note that (1.2) admits semi-positive solutions which have at least one component being zero.

Solutions $\vec{u} = (u_1, \ldots, u_N) \in [H^1(\mathbb{R}^n)]^N$ correspond to critical points of the energy functional associated with (1.2)

$$E(u_1, \ldots, u_N) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^{N} (|\nabla u_j|^2 + \lambda_j u_j^2) - \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j=1}^{N} \beta_{ij} u_i^2 u_j^2. \tag{1.3}$$

Since we assume $n \leq 3$, the Sobolev embedding implies that E is well-defined and of class C^2. A solution is said to be a *ground state* if it has the least energy among all nontrivial solutions. We investigate the existence of a positive ground state solution and focus on the case of $\beta_{ij} > 0$ in this note. Our results here complement those in an earlier paper [1] in which Morse theory (e.g. [2])is used to show the existence of bound states.

Clearly, if there exists a ground state solution then there also exists a semi-positive one. Let $H_r^1(\mathbb{R}^n)$ consist of all radially symmetric functions in $H^1(\mathbb{R}^n)$, and set $X := [H^1(\mathbb{R}^n)]^N$ and $X_r := [H_r^1(\mathbb{R}^n)]^N$. X is a Hilbert space with inner product

$$\langle \vec{u}, \vec{v} \rangle = \sum_{j=1}^{N} \langle u_j, v_j \rangle_{H^1}$$

and X_r is a Hilbert subspace.

Theorem 1.1 (1.2) *has a semi-positive ground state $\vec{u} \in X_r$. It is of mountain pass type and has Morse index 1 considered as critical point of E on X and on X_r.*

Proof We define the non-negative functionals $\mathcal{A}, \mathcal{B} : X \to \mathbb{R}$ by

$$\mathcal{A}(\vec{u}) = \int_{\mathbb{R}^n} \sum_{j=1}^{N} (|\nabla u_j|^2 + \lambda_j u_j^2)$$

and

$$\mathcal{B}(\vec{u}) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{N} \beta_{ij} u_i^2 u_j^2,$$

so that $E(\vec{u}) = \frac{1}{2}\mathcal{A}(\vec{u}) - \frac{1}{4}\mathcal{B}(\vec{u})$. Now we consider the Nehari manifold

$$\begin{aligned} M &= \{\vec{u} \in X \setminus \{0\} \mid E(\vec{u}) = \max_{t>0} E(t\vec{u})\} = \{\vec{u} \in X \setminus \{0\} \mid E'(u)u = 0\} \\ &= \{\vec{u} \in X \setminus \{0\} \mid \mathcal{A}(\vec{u}) = \mathcal{B}(\vec{u})\} \end{aligned}$$

and the radial Nehari manifold $M_r := M \cap X_r$. All nontrivial critical points of E have to be on M. A ground state is a minimizer of E on M. We have to show that the minimizer of E on M is achieved by a semi-positive $\vec{u} \in M_r$. As a critical point of $E|_M$ it is a critical point of the full functional E, hence a solution of (1.2). Since M is a codimension 1 manifold, the Morse index of any minimizer of E on M