J. Partial Diff. Eqs. **19**(2006), 48–79
 Vol.19, No.1

WELL–POSEDNESS OF A FREE BOUNDARY PROBLEM IN THE LIMIT OF SLOW–DIFFUSION FAST–REACTION SYSTEMS*

Chen Xinfu and Gao Congyu
(Department of Mathematics University of Pittsburgh, Pittsburgh, PA 15260)
(E-mail: xinfu@pitt.edu, cogst1@pitt.edu)
(http://www.pitt.edu/~xinfu, www.pitt.edu/~cogst1)

Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

(Received Nov. 25, 2005)

Abstract We consider a free boundary problem obtained from the asymptotic limit of a FitzHugh–Nagumo system, or more precisely, a slow–diffusion, fast–reaction equation governing a phase indicator, coupled with an ordinary differential equation governing a control variable v. In the range $(-1, 1)$, the v value controls the speed of the propagation of phase boundaries (interfaces) and in the mean time changes with dynamics depending on the phases. A new feature included in our formulation and thus made our model different from most of the contemporary ones is the nucleation phenomenon: a phase switch occurs whenever v elevates to 1 or drops to -1. For this free boundary problem, we provide a weak formulation which allows the propagation, annihilation, and nucleation of interfaces, and excludes interfaces from having (space–time) interior points. We study, in the one space dimension setting, the existence, uniqueness, and non–uniqueness of weak solutions. A few illustrating examples are also included.

Key Words Well-posedness; FitzHugh-Nagumo system; free boundary problem.
2000 MR Subject Classification 35R35.
Chinese Library Classification O175.29, O175.26.

1. Introduction

Interfacial phenomena are commonplace in physics, chemistry, biology, and in various other fields. They occur whenever a continuum is present that can exist in at least two different "states" and there is some mechanism that generates or enforces a spatial separation between these two states. The common boundaries are called *interfaces* or *free boundaries*. These interfaces are observed to manifest various geometrical patterns, such as front of shock waves [1, 2], rotating spiral waves and expanding target patterns [3, 4]. From both physical and mathematical point of view, it is very important to know the shape and motion of these boundaries.

*This research is partially supported by the National Science Foundation Grant DMS–9971043.

One of the commonly used mathematical model in studying the interfacial phenomena is the following reaction diffusion system:

$$\begin{cases} u_t = \varepsilon\Delta u + \varepsilon^{-1} f(u,v), \\ v_t = D\Delta v + g(u,v) \end{cases} \tag{1.1}$$

with typical f and g given by

$$f(u,v) = F(u) - v, \quad F(u) = u(3/\sqrt[3]{2} - 2u^2), \quad g(u,v) = u - \gamma v - b, \tag{1.2}$$

where $D \geq 0$, $\gamma > 0$ and $b \in \mathbb{R}$ are constants, and $0 < \varepsilon \ll 1$ serves as a small parameter. This system models the propagation of chemical waves in excitable or bistable or oscillatory media, where u and v represent the propagator and controller respectively [5]. It also describes an activator-inhibitor model; see Ohta, Mimura and Kobayashi [6]. When $D = O(\varepsilon)$, (1.1) was used by Tyson and Fife to study the Belousov-Zhabotinskii reagent [7]. When $D = 0$, (1.1) is the well–known FitzHugh-Nagumo model for nerve impulse propagation; see [8, 9, 10, 11, and references therein].

In this paper, we shall study a free boundary problem obtained as the singular limit, as $\varepsilon \searrow 0$, of the FitzHugh–Nagumo model. For the reader's convenience, here we provide a formal derivation of this free boundary problem; for more details, see Fife [5, Chapter 4], X.Y. Chen [12], and X. Chen [13].

The local minimum and maximum of the cubic function $F(u)$ in (1.2) is -1 and 1. If $v \in (-1,1)$, the equation $f(u,v) = F(u) - v = 0$, for u, has three real roots, $h_-(v)$, $h_0(v)$ and $h_+(v)$, where $h_-(v) < h_0(v) < h_+(v)$; see Figure 1 (c). The roots $h_-(v)$ and $h_+(v)$ are stable equilibria of the ode

$$u_t = \varepsilon^{-1} f(u,v), \tag{1.3}$$

with attraction domains $(-\infty, h_0(v))$ and $(h_0(v), \infty)$ respectively. If $\mp v > 1$, $u = h_\pm(v)$ is the only equilibrium of (1.3) and is globally stable; i.e., its attraction domain is $\mathbb{R}$.

Consider (1.1a), regarding v as a known function. Since ε is small, (1.1a) is often referred to as a *slow-diffusion fast-reaction* equation[5, 14]. For smooth initial data $u(x,0)$, the $\varepsilon\Delta u$ term can be neglected initially, and (1.1a) can be approximated by (1.3). Hence, at each point x in the spatial domain, u approaches quickly to either $h_+(v(x,0))$ or $h_-(v(x,0))$ depending on the sign of $u(x,0) - h_0(v(x,0))$ (extending $h_0(v) = \pm\infty$ for $\pm v > 1$). Consequently, two disjoint spatial regions Ω_+ and Ω_-, where $u \approx h_+(v)$ and $u \approx h_-(v)$ respectively, are formed. The remaining region $\Omega_0 = (\Omega_+ \cup \Omega_-)^{\mathrm{c}}$, located near the set where $u(x,0) - h_0(v(x,0)) = 0$, is very thin and can be regarded as a hypersurface called *interface*. This process is commonly referred to as *the generation of interface*. A rigorous verification of this process for the one space dimensional case was first carried out by Fife and Hsiao [15]. In the special case $v \equiv 0$ (and in general space dimension), de Mottoni and Schatzman [16] established a