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STABILITY AND INSTABILITY OF SOLITARY WAVES FOR ABSTRACT COMPLEX HAMILTONIAN SYSTEM*

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Abstract This paper is concerned with the orbital stability and orbital instability of solitary waves for some complex Hamiltonian systems in abstract form. Under some assumptions on the spectra of the related operator and the decaying estimates of the semigroup, the sufficient conditions on orbital stability and instability are obtained.

Key Words Complex Hamiltonian systems; solitary wave; orbital stability; orbital instability.

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1. Introduction

It is well known that a large number of Hamiltonian systems can be written in the following form

$$\frac{du}{dt} = JE'(u(t)) \tag{1.1}$$

here J is a skew-symmetric linear operator and E is a functional (the energy). There are many prototypical examples of (1.1), such as the generalized Kdv equation [1], wave equation $u_{tt} - u_{xx} + f(u) = 0$, and Schrödinger equation etc. For solitary waves $\phi(x - \omega t)$, $e^{-i\omega t}\phi(x)$ of system (1.1), Statah, etc. [2], proved that they are orbitally stable if the scalar function $d(\omega) = E(\phi_\omega) + \omega Q(\phi_\omega)$ satisfies $d''(\omega) > 0$, while they are orbitally unstable when $d''(\omega) < 0$ and J is onto. As a continuation of [2] some abstract stability/instability results for more general types of solitary waves were obtained in [3], which includes the solitary waves $e^{-i\omega t}\phi(x - vt)$, with ω and v varying in some intervals.

It is worth noting that in the proof of instability theorems in [2, 3] the operator J is required to be onto, which implies that the abstract instability results in [2, 3] can't be

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applied directly to the system (1.1) when J is not onto, such as KdV, BBM equations etc. Based on the idea of [2], by detailed decaying estimates, it was proved in [1] that the solitary waves of the generalized KdV equation are orbitally unstable if $d''(v) < 0$. Using the idea of proof in [1], the instability of solitary waves for BBM equation, Boussinesque equation was also studied [4 - 6]. Recently, for some nonlinear complex BBM equations, the orbital instability of a family of solitary waves with rotation was obtained [7]. The instability of solitary waves for a complex Boussinesque equation and nonlinear composite media system was investigated in [8].

Motivated by these results, in this paper we consider the abstract complex Hamiltonian system coupled by two complex equations

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}(t)) \tag{1.2}$$

where $\vec{u} = \{u, v\}^t = \{u_1 + iu_2, v_1 + iv_2\}^t$. We investigate the stability and instability of a family of solitary waves with rotation, precisely speaking in the form of $e^{-i\varphi}\vec{\zeta}(x - vt)$, where v can exist in some interval and φ can be any value of parameter. For the solitary waves $e^{-i\omega t}\phi(x-vt)$, with $\omega \in (\omega_1, \omega_2)$ and $v \in (v_1, v_2)$, the abstract stability/instability results [3] can't be applied to the case when rotation speed ω is just zero.

Separating the real part of $\vec{u}$ from the imaginary part, we can rewrite (1.2) in the following nonlinear real systems coupled by four equations

$$\frac{d\vec{w}}{dt} = JE'(\vec{w}(t)) \tag{1.3}$$

$\vec{w} = \{u_1, u_2, v_1, v_2\}^t$ is an unknown real valued vector function. In the following of this paper we just consider the stability/instability of real solitary waves of the system (1.3). Note that the complex solitary wave $e^{-i\varphi}\vec{\zeta}(x - vt)$ of (1.2) correspond to the real solitary wave of (1.3) in the form of

$$\exp(\mathcal{A}\varphi)\vec{\phi}(x - vt) \triangleq \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix} (x - vt)$$

and $\mathcal{A} = \text{diag}(a, a)$ is the block diagonal 4×4 matrix with the block $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is easy to check that for linear operator $H \triangleq E''(\vec{\phi}_v) + vQ''(\vec{\phi}_v)$ of the system (1.2), the geometric multiplicity of zero eigenvalue is at least two, which are induced by translation and rotation symmetry of the complex system (1.2), thus the orbital stability/instability of the solitary waves with rotation should be in an appropriate weak sense. In fact for the complex system the orbital instability of real solitary waves under real perturbations does not imply the orbital instability of the corresponding