J. Partial Diff. Eqs. **18**(2005), 322–326
©International Academic Publishers Vol.18 No.4

A NOTE ON THE YAMABE PROBLEM*

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(Received Jul. 20, 2005)

Dedicated to the memory of Professor S.S. Chern

Abstract In this note we verify the key inequality $Y_1(\mathcal{M}) < Y_1(S^n)$ for the Yamabe constant $Y_1(\mathcal{M})$ for manifolds $\mathcal{M}$ not conformal to the unit sphere, by using a solution to an associated equation as a test function.

Key Words Convex hypersurface; Weingarten curvature.
2000 MR Subject Classification 35J60, 53C45.
Chinese Library Classification O175.29, O175.25.

Let $(\mathcal{M}, g_0)$ be a compact, smooth, orientable Riemannian manifold of dimension $n \geq 3$. Denote by $R = R_g$ the scalar curvature and by $[g_0] = \{g \mid g = u^{4/(n-2)}g_0\}$ the set of metrics conformal to g_0. The Yamabe problem is to find a solution to the equation

$$\Box u =: -\Delta u + \frac{n-2}{4(n-1)} R u = \lambda u^{\frac{n+2}{n-2}}, \tag{1}$$

where λ is a constant, which is positive if the Yamabe constant

$$Y_1(\mathcal{M}) = \inf_{g \in [g_0]} Q(g)$$

is positive. We denote

$$Q(g) = \int_{\mathcal{M}} R_g \, d\mu_g / V_g^{(n-2)/n},$$

where V_g is the volume of $(\mathcal{M}, g)$ and $d\mu_g$ is the volume element. Based on the earlier works [1-3], the Yamabe problem was finally solved in [4]. A key ingredient is to verify

$$Y_1(\mathcal{M}) < Y_1(S^n) \tag{2}$$

for manifolds with positive Yamabe constant which are not conformal to the unit sphere S^n with standard metric.

*Supported by the Natural Science Foundation of China and the Australian Research Council.

Inequality (2) was verified by Aubin [1] for manifolds of dimensions $n \geq 6$ not locally conformally flat, and by Schoen [4] for the remaining cases. The proofs in [1, 4] were unified in [5] by introducing the conformal normal coordinates. In this note we verify the inequality (2) by using a solution to an associated equation (see (11) below) rather than the usual test function. The advantage of this proof is that it also applies to the Yamabe problem for higher order curvatures [6], where it is complicated to construct an explicit admissible test function.

For a given point $0 \in \mathcal{M}$, choose a conformal metric, which we may assume is g_0 itself [5, 7], such that in the normal coordinate of g_0, $R(0) = 0$, $\nabla R(0) = 0$ and

$$\Delta R(0) = -\frac{1}{6}|W(0)|^2, \tag{3}$$

$$\det (g_0)_{ij} \equiv 1 \quad \text{near} \;\; 0, \tag{4}$$

where $W(0)$ is the Weyl tensor at 0. For formula (4), see [8, 9] or p.159 of [7]. Denote by r the geodesic distance from x to 0, and B_ρ the geodesic ball with center 0 and radius ρ. Since the Yamabe constant $Y_1(\mathcal{M})$ is positive, the Green function G at 0 is unique, that is G is the unique solution to the equation

$$\Box G = (n-2)\omega_{n-1}\delta_0, \tag{5}$$

where δ_0 is Dirac measure at 0, and ω_{n-1} is the area of the sphere S^{n-1}.

Lemma 1[5] *Suppose $\mathcal{M}$ is not conformal to the unit sphere S^n. Then the Green function G has the asymptotic behavior*

$$G(x) = r^{2-n} + \zeta(x), \tag{6}$$

where $\zeta = o(r^{2-n})$ is a function satisfying

$$\zeta \geq A - c_0\, R_{,ij}(0) x_i x_j r^{4-n} \quad \textit{near} \;\; 0, \tag{7}$$

where A is a positive constant, $c_0 = 0$ if $n \leq 6$ or $\mathcal{M}$ is locally conformally flat; and c_0 is a positive constant in the remaining cases (in this case we assume $|W(0)| \neq 0$).

A more precise estimate of ζ can be found in [5], see also [10]. Recently Christ and Lohkamp announced a proof of the positive mass theorem for all dimensions. Hence one can choose $c_0 = 0$ in (7). But we will not use it here. Now let

$$v_\varepsilon(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + r^2}\Big)^{\frac{n-2}{2}},$$

where ε is a small positive constant. Note that in a normal coordinate, under condition (4), the Laplacian Δ on $\mathcal{M}$ is equal to the Euclidean Laplacian when applying to functions of r alone, hence we have

$$-\Delta v_\varepsilon = n(n-2) v_\varepsilon^{\frac{n+2}{n-2}} \quad \text{in} \;\; B_{\rho_0} \tag{8}$$