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SOME ENTROPY INEQUALITIES FOR A QUASILINEAR DEGENERATE HYPERBOLIC EQUATION*

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Abstract The aim of this paper is to discuss some degenerate hyperbolic equation

$$u_t + \varphi(u)_x = 0,$$

where $\varphi \in C^1(\mathbb{R} \setminus \{0\}) \cap C^2(\mathbb{R} \setminus \{0\})$ is a nondecreasing function in $\mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$. Some entropy inequalities are obtained and can be applied to study the existence of local BV solutions of the above equation with local finite measures as initial conditions.
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1. Introduction

In this paper we consider the following quasilinear degenerate hyperbolic equation of the form

$$u_t + \varphi(u)_x = 0 \tag{1.1}$$

in $Q_T \equiv \mathbb{R} \times (0, T)$ with the following initial condition

$$u(x, 0) = u_0(x) \tag{1.2}$$

for all $x \in \mathbb{R}$, where φ is a continuons nondecreasing function in $\mathbb{R}$, $0 \le u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and $\mathbb{R} = (-\infty, +\infty)$.

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By $BV(Q_T)$, we mean the class of all functions in Q_T of locally bounded variation. In other words, $u \in BV(Q_T)$ if and only if $u \in L^1_{loc}(Q_T)$ and u_t and u_x are regular measures in Q_T of locally bounded variation. By $BV(\mathbb{R})$, we mean the class of all functions in $\mathbb{R}$ of locally bounded variation.

We consider BV solutions of (1.1)-(1.2) in the following sense.

Definition 1.1 *A nonnegative function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is said to be a solution of* (1.1) *and* (1.2)*, if u satisfies the following conditions* [H1] *and* [H2]:

[H1] *For any $\xi \in C_0^\infty(Q_T)$ with $\xi \geq 0$, we have*

$$\int\int_{Q_T} \operatorname{sign}(u-k)[(u-k)\xi_t + (\varphi(u) - \varphi(k))\xi_x]dxdt \geq 0$$

for all $k \in \mathbb{R}$, where

$$\operatorname{sign}(u-k) = \begin{cases} 1, & \text{if} \quad u > k, \\ 0, & \text{if} \quad u = k, \\ -1, & \text{if} \quad u < k. \end{cases}$$

[H2] *For any $\eta \in C_0^\infty(\mathbb{R})$, we have*

$$\operatorname{ess}\lim_{t \to 0^+} \int_{\mathbb{R}} \eta(x) u(x,t) dx = \int_{\mathbb{R}} \eta(x) u_0(x) dx.$$

It is known that the Cauchy problem (1.1)-(1.2) has a unique solution $u \in L^\infty(Q_T) \cap BV(Q_T)$, see J. Smoller [1].

Our main results are

Theorem 1.1 *Assume that $\varphi \in C^1(\mathbb{R} \setminus \{0\}) \cap C^2(\mathbb{R} \setminus \{0\})$ is a continuous non-decreasing function in $\mathbb{R}$, $\varphi(0) = 0$, and*

$$\varphi'(s) > 0, \qquad \frac{\varphi(s)\varphi''(s)}{\varphi'^2(s)} \geq K_1, \qquad \frac{\varphi}{\varphi'} \in C(\mathbb{R}) \tag{1.3}$$

for all $s \in \mathbb{R} \setminus \{0\}$, where K_1 is a given positive constant. If u is a solution of the Cauchy problem (1.1)-(1.2)*, then we have*

$$u_t \geq -\frac{1}{K_1 t} \cdot \frac{\varphi(u)}{\varphi'(u)} \tag{1.4}$$

in the sense of distribution.

Remark 1.1 Clearly, the function $\varphi(s)$ satisfying (1.3) has a typical example as follows

$$\varphi(s) = |s|^{m-1}s$$

for all $s \in \mathbb{R}$, where $m > 1$.

In addition, we also have