J. Partial Diff. Eqs. **18**(2005), 1–12
©International Academic Publishers Vol.18 No.1

BUBBLES OF LANDAU-LIFSHITZ EQUATIONS WITH APPLIED FIELDS

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(Received Jun. 28, 2003; revised Mar. 8, 2004)

Abstract In this paper, we discuss the Landau-Lifshitz equations with applied magnetic fields. The equations describing the bubbles in the ferromagnets and the behaviors of the solutions near the singularities are given. We found that the applied fields do not affect the bubbles and we have the same conclusions as in reference [1].
Key Words Landau-Lifshitz equations; magnetic fields; bubbles.
2000 MR Subject Classification 35J55, 35Q40.
Chinese Library Classification O175.29

1. Introduction

Let M be a two dimensional manifold without boundary. We consider the following Landau-Lifshitz equation describing the evolution of spin fields in continuum ferromagnets with applied magnetic fields:

$$\partial_t u = -u \times (u \times \Delta u) + u \times \Delta u + u \times h(u), (x,t) \in M \times (0,+\infty) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{1.2}$$

where $|u_0(x)| = 1$ for $x \in \Omega$, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the spin chain vector and $h(u)$ denotes the applied fields.

Using $|u| = 1$ and $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we know that (1.1) is equivalent to

$$\partial_t u = \Delta u + u \times \Delta u + |\nabla u|^2 u + u \times h(u), \quad (x,t) \in M \times (0,+\infty). \tag{1.3}$$

*The first author is supported by the Natural Science Foundation of China (No.19971030, No.10471050) and the Natural Science Foundation of Guangdong Province(No.000671, No.031495).

This equation was first derived on phenomenological grounds by Landau-Lifshitz [2]. It plays a fundamental role in the understanding of non-equilibrium magnetism.

Let the applied field $h(u)$ satisfy: $h(u) \in L^\infty(W^{1,\infty}(M), \mathbb{R}_+)$. Then for any $u_0 \in H^1(M, S^1)$, the Cauchy problem (1.1) and (1.2) admits unique solution [3] which is regular within finite time and develops singular points beginning at some time, $t = T$ for example, but with at most finitely many points on the plane $t = T$.

In this note, we want to know what will happen near these points and what is the local behavior of the solution near its singularities.

For the solutions of harmonic map heat flow, Struwe [4] has shown that for any $u_0 \in H^1(M, N)$, the solution exists and is unique which is smooth away from at most finitely many points in $M \times \mathbb{R}_+$. Moreover, if the solution u develops a singularity at (x_0, T), by choosing a suitable sequence $t_i \uparrow T$ and rescaling $u(\cdot, t_i)$ properly near x_0, one can obtain finitely many nonconstant harmonic maps ϕ_i $(1 \le i \le L)$ from $\mathbb{R}^2 \to N$ and they can be extended to the harmonic maps from S^2 to N referred as bubbles. Qing [5] proved that if the target manifold N is a sphere, these bubbles are responsible for the energy loss at the singular times $T = \infty$. This is the so called energy identity:

$$\lim_{t_i \uparrow T} \int_{B_\delta(x_0)} |\nabla u|^2(x, t_i)dx = \int_{B_\delta(x_0)} |\nabla u|^2(x, T)dx + \sum_{i=1}^{L} \int_{\mathbb{R}^2} |\nabla \phi_i|^2(x)dx \tag{1.4}$$

where $B_\delta(x_0)$ is a small neighborhood of x_0, which does not contain any other singular points of u. Recently Qing's results have been generated to the flow of harmonic maps to arbitrary compact target manifold in [6–8] In [7] the energy identity (1.4) has been proved for any general target manifolds and for any finite singular time $T < \infty$.

For the ferromagnetic equation without applied fields, [1] proves the similar results as above. However, in our case, as stated in [3], since we do not know whether the energy is decreasing with time, we proved in [3] that the singular points the solution develops at time T may keep singular with time increasing, the exception set of singular points in $M \times \mathbb{R}_+$ may be not a finite set but some lines. So our discussions only apply to the first time $t = T$ at which the solution first develops singular points. Our studies show that the bubbles at layer $t = T$ can be described by the same method as in [1].

In this paper, the following notations are used. For a point $z_0 = (x_0, t_0)$, $P_r(z_0)$ denotes the cylinder

$$P_r(z_0) = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ : \quad |x - x_0| < r,\ t_0 - r^2 < t < t_0\}$$

and $B_r(x_0)$ denotes the ball centered at $x = x_0$ with radius r. If $z_0 = (0, 0)$ or $x_0 = 0$, we simply denote $P_r = P_r(0)$ and $B_r = B_r(0)$. For $Q \subset \mathbb{R}_+ \times \mathbb{R}^2$, $C^{\alpha,2\alpha}(Q)$ denotes the Hölder space on Q and

$$W_p^{1,2} = \{u \in L^p(Q) : \ u_t, \nabla u, \nabla^2 u \in L^p(Q)\}.$$

The energy of $u(x, t)$ on $\Omega \subset \mathbb{R}^2$ at the time t is denoted by $E(u, \Omega)(t)$ i.e.

$$E(u, \Omega)(t) = \int_\Omega e(u(x, t))dx \tag{1.5}$$