SELF-SIMILAR SINGULAR SOLUTION OF A P-LAPLACIAN EVOLUTION EQUATION WITH GRADIENT ABSORPTION TERM*

Shi Peihu
(Department of Mathematics, Southeast University, Nanjing 210096, China)
(E-mail: sph2106@yahoo.com.cn)
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Abstract In this paper we deal with the self-similar singular solution of the p-Laplacian evolution equation \( u_t = \text{div}(\|\nabla u\|^{p-2}\nabla u) - |\nabla u|^q \) for \( p > 2 \) and \( q > 1 \) in \( \mathbb{R}^n \times (0, \infty) \). We prove that when \( p > q + n/(n+1) \) there exist self-similar singular solutions, while \( p \leq q + n/(n+1) \) there is no any self-similar singular solution. In case of existence, the self-similar singular solutions are the self-similar very singular solutions, which have compact support. Moreover, the interface relation is obtained.

Key Words p-Laplacian evolution equation; gradient absorption; self-similar; singular solution; very singular solution.

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1. Introduction and Main Results

In this paper we consider the self-similar singular solution of the p-Laplacian evolution equation with nonlinear gradient absorption term

\[ u_t = \text{div}(\|\nabla u\|^{p-2}\nabla u) - |\nabla u|^q \text{ in } \mathbb{R}^n \times (0, \infty), \tag{1.1} \]

where \( p > 2 \) and \( q > 1 \). Here by singular solution we mean a nonnegative and nontrivial solution \( u(x, t) \), which is continuous in \( \mathbb{R}^n \times [0, \infty) \) \( \setminus \{0, 0\} \) and satisfies

\[ \lim_{t \to 0} \sup_{|x| > \varepsilon} u(x, t) = 0, \quad \forall \ \varepsilon > 0. \tag{1.2} \]

A singular solution \( u(x, t) \) is called a very singular solution provided that it satisfies

\[ \lim_{t \to 0} \int_{|x| < \varepsilon} u(x, t) \, dx = \infty, \quad \forall \ \varepsilon > 0. \tag{1.3} \]

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By self-similar solution we mean that the solution \( u(x, t) \) has the following form
\[
    u(x, t) = \left( \frac{\alpha}{t} \right)^\alpha f(|x| \left( \frac{\alpha}{t} \right)^{\alpha \beta}), \quad \alpha = \frac{p - q}{2q - p}, \quad \beta = \frac{q + 1 - p}{p - q}.
\] (1.4)

To guarantee the constants \( \alpha \) and \( \beta \) are positive, here we consider the following case
\[
    2q > p > q, \quad q + 1 - p > 0.
\] (1.5)

Consequently, the self-similar singular solution to (1.1), if it exists, satisfies the following ODE boundary problem
\[
    \begin{cases}
        (|f'|^{p-2}f')' + \frac{n-1}{r} |f'|^{p-2}f' + \beta r f' + f - |f'|^q = 0, \\
        f(0) = a > 0, \quad \lim_{r \to \infty} r^{1/\beta} f(r) = 0,
    \end{cases}
\] (1.6)

where \( f = f(r) \) is the function of self-similar variable \( r = |x| \left( \frac{\alpha}{t} \right)^{\alpha \beta} \), the prime denotes the differentiation with respect to \( r \).

In this paper we set
\[
    \nu = p + (p - 2)/\beta = q + (q - 1)/\beta > 2.
\]

Singular solutions were first discovered for the semilinear heat equation
\[
    u_t = \Delta u - u^p.
\] (1.7)

Brezis and Friedman [1] in 1983 proved that (1.7) admits a unique singular solution for every \( c \in (0, \infty) \) when \( 1 < p < 1 + 2/n \) such that \( \lim_{t \to 0} \int_{|x|<\varepsilon} u(x, t) dx = c, \quad \forall \varepsilon > 0 \), which is called a fundamental solution with initial mass \( c \), while it has no such solutions for \( p \geq 1 + 2/n \). Shortly, Brezis, Peletier and Terman [2] had proved that (1.7) possesses a unique very singular solution when \( 1 < p < 1 + 2/n \). Since that time many authors studied the self-similar singular solutions (see [3-8] and the references therein) of the following equations
\[
    \begin{align*}
    &u_t = \Delta (u^m) - u^p, \quad 0 < m < \infty, \quad p > 1, \\
    &u_t = \Delta (u^m) - |\nabla u|^p, \quad 1 \leq m < \infty, \quad p > 1, \\
    &u_t = \text{div}(|\nabla u^m|^{p-2}\nabla u^m) - u^q, \quad 0 < m < \infty, \quad p > 1, \quad q > 1.
    \end{align*}
\]

In addition, large time behavior of solutions to the Cauchy problems of the above equations with absorption \( u^p \) (or \( u^q \)) can be characterized by their very singular, self-similar solutions and fundamental solutions, see [9-14].

To study the boundary value problem (1.6), we consider the initial problem
\[
    \begin{cases}
        (|f'|^{p-2}f')' + \frac{n-1}{r} |f'|^{p-2}f' + \beta r f' + f - |f'|^q = 0, \quad r > 0, \\
        f(0) = a > 0, \quad f'(0) = 0.
    \end{cases}
\] (1.8)

Let \( f(r; a) \) be the solution of (1.8) and \( (0, R(a)) \) be the maximal existence interval, where \( f(r; a) > 0 \). Our main results read as follows: