J. Partial Diff. Eqs. **17**(2004), 316–328
 Vol.17, No.4

THE CAUCHY PROBLEM FOR SOME DISPERSIVE WAVE EQUATIONS

Zhang Wenling
(National Natural Science Foundation of China, Beijing 100085, China)
(E-mail: gmwei@mail.ustc.edu.cn)
(Received Feb. 23, 2004)

Abstract In this paper, we consider the Cauchy problem for some dispersive equations. By means of nonlinear estimate in Besov spaces and fixed point theory, we prove the global well-posedness of the above problem. What's more, we improve the scattering result obtained in [1].
Key Words Well-posedness; Cauchy problem; dispersive equation; scattering.
2000 MR Subject Classification 35Q30.
Chinese Library Classification O175.29.

1. Introduction and Main Results

We consider the well-posedness and scattering theory to the Cauchy problem for nonlinear dispersive equation as in [1]

$$\begin{cases} Mu_t + u_x + f(u)_x = 0, \\ u(0) = \varphi(x), \end{cases} \tag{1.1}$$

or more generally, of its multi-dimensional case

$$\begin{cases} Mu_t + (b, \nabla)u + (\nabla, \vec{f}(u)) = 0, \\ u(0) = \varphi(x), \end{cases} \tag{1.2}$$

where $\vec{f}(u) \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ satisfies

$$\vec{f}(0) = 0, \quad |\vec{f}(u)| \le C|u|^{p+1}, \quad 0 < p < \infty. \tag{1.3}$$

and M is a pseudo-differential operator with

$$\widehat{Mu}(\xi) = m(\xi)\hat{u}(\xi), \tag{1.4}$$

and the symbol $m(\xi)$ satisfies the following assumption:

(H1) $m(\xi) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^+)$, and there exist positive constants C_1, C_2 such that

$$0 < C_1(1 + |\xi|)^\mu \le m(\xi) \le C_2(1 + |\xi|)^\mu, \quad \mu > 1.$$

In special, for $M = I - \partial_x^2$, (1.1) coincides with BBM equation

$$u_t + u_x - u_{xxt} + f(u)_x = 0, \tag{1.5}$$

which is originated from the wave equation

$$u_t + u_x = 0, \tag{1.6}$$

by introducing dispersive effect u_{xxt} and adding nonlinear term $f(u)$, but when it comes to u_{xxx}, we get generalized KdV equation

$$u_t + u_x + u_{xxx} + f(u)_x = 0. \tag{1.7}$$

Both kinds of equations in arbitrary space dimensions have been extensively studied by many authors [2 - 6]. Their main results can be stated as the following:

For any bounded smooth domain Ω and $\max(1, \frac{n}{2}) < p < \infty$, there is a unique global strong solution for the Initial-Boundary-Value problem of GBBM equation (1.2) in $\mathcal{W}^{2,p}(\Omega)$; for any unbounded smooth domain Ω or $\Omega = \mathbb{R}^n, \max(1, \frac{n}{2}) < p < n$, there exist a unique global strong solution for the IBV problem or Cauchy problem. The paper [7] established the existence and uniqueness of global strong solution for the IBV problem and Cauchy problem of inhomogeneous GBBM equation in $\mathcal{W}^{2,p}(\Omega)$ in the case of $\max(1, \frac{n}{2}) < p < \infty$

In this paper, we pay our attention to (1.1) and (1.2) and our main results can be expressed as follows:

Theorem 1.1 (1) *If $f(u)$ satisfies (1.3) and $m(\xi)$ satisies (H1) for $n = 1$, then (1.1) or its associated integral equation*

$$u(t) = \mathcal{S}(t)\varphi - \int_0^t \mathcal{S}(t-\tau)M^{-1}\partial_x f(u(\tau))d\tau, \tag{1.8}$$

has a strong solution $u(t) \in \mathcal{C}(\mathbb{R}; H^{\frac{\mu}{2}})$, where

$$\mathcal{S}(t)\varphi = e^{-tM^{-1}\partial_x}\varphi = \mathcal{F}^{-1}e^{-t\frac{i\xi}{m(\xi)}}\mathcal{F}\varphi. \tag{1.9}$$

(2) *If $\varphi \in H^s$ for $\frac{\mu}{2} < s < p + \mu - \frac{1}{2}$, then we have $u(t) \in \mathcal{C}(\mathbb{R}; H^s)$.*

Theorem 1.2 (1) *If $\vec{f}(u)$ satisfies (1.3) and $m(\xi)$ satisfies (H1) for $n \geq 2$, then(1.2) or its associated integral equation*

$$u(t) = \mathcal{S}(t)\varphi - \int_0^t \mathcal{S}(t-\tau)M^{-1}(\nabla, f(u(\tau)))d\tau \tag{1.10}$$

has a strong solution $u(t) \in \mathcal{C}(\mathbb{R}; H^{\frac{\mu}{2}})$ if either of the following holds:

(i) $0 < p < \dfrac{2}{n}, \dfrac{np+2}{p+2} < \mu < \dfrac{np+2}{2}$;