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# CAUCHY PROBLEM FOR QUASILINEAR HYPERBOLIC SYSTEMS WITH CHARACTERISTICS WITH CONSTANT MULTIPLICITY

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**Abstract** For quasilinear hyperbolic systems with characteristics of constant multiplicity, suppose that characteristics of constant multiplicity($> 1$) are linearly degenerate, by means of generalized normalized coordinates we get the global existence and the blow-up phenomenon of the $C^1$ solution to the Cauchy problem under an additional hypothesis.

**Key Words** Quasilinear hyperbolic system; Cauchy problem; global classical solution; formation of singularity; life span.

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## 1. Introduction

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0, \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$ is the unknown vector function of $(t,x)$ and $A(u)$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j=1,\cdots,n)$.

By hyperbolicity, for any given $u$ on the domain under consideration, $A(u)$ has $n$ real eigenvalues $\lambda_1(u),\cdots,\lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i=1,\cdots,n$, let $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))$ (resp. $r_i(u)=(r_{i1}(u),\cdots,r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u) \qquad (\text{resp.} \quad A(u)r_i(u)=\lambda_i(u)r_i(u)). \tag{1.2}$$

We have

$$\det|l_{ij}(u)|\neq 0 \qquad (\text{equivallently}, \quad \det|r_{ij}(u)|\neq 0). \tag{1.3}$$

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u)\equiv\delta_{ij} \quad (i,j=1,\cdots,n) \tag{1.4}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{1.5}$$

where $\delta_{ij}$ stands for the Kronecker's symbol.

For quasilinear hyperbolic systems with characteristics of constant multiplicity, without loss of generality, we suppose that, in a neighbourhood of $u = 0$,

$$\lambda(u) \stackrel{\triangle}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{1.6}$$

where $1 \leq p \leq n$. When $p = 1$, the system (1.1) is strictly hyperbolic; while, when $p > 1$, the system (1.1) is a non-strictly hyperbolic system with characteristics of constant multiplicity $p(> 1)$. All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)(i, j = 1, \cdots, n)$.

For the Cauchy problem of the system (1.1) with the following initial data

$$t = 0 : u = \phi(x), \tag{1.7}$$

where $\phi(x) \in C^1$ is a vector function satisfying

$$\theta \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}}\{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty, \tag{1.8}$$

in which $\mu > 0$ is a constant, Li, Zhou and Kong have obtained the global existence and the blow-up phenomenon of the classical solution to the Cauchy problem for the strictly hyperbolic system and for the hyperbolic system of conservation laws with characteristics of constant multiplicity in [1] and [2] respectively. The hyperbolic system of conservation laws with characteristics of constant multiplicity possesses normalized coordinates (see [2]), while, for general quasilinear hyperbolic systems, we do not know if there exist normalized coordinates. To eliminate the use of normalized coordinates, Li and Kong have obtained the corresponding results in [3] under the hypotheses that all characteristics of constant multiplicity($> 1$) are linearly degenerate and all simple characteristics are either genuinely nonlinear or linearly degenerate. In this paper, suppose that all characteristics of constant multiplicity ($> 1$)are linearly degenerate, namely, when $p > 1$, for any given $u$ on the domain under consideration,

$$\nabla\lambda(u)r_i(u) \equiv 0, \quad \forall i \in \{1, \cdots, p\}, \tag{1.9}$$

we consider the Cauchy problem by means of generalized normalized coordinates and obtain the global existence and the blow-up phenomenon of the $C^1$ solution under an additional hypothesis. Thus, in some sense, we unify the corresponding results in [2] and [3].

As we know, the $i$-th ($i \in \{p + 1, \cdots, n\}$) characteristic $\lambda_i(u)$ is linearly degenerate if for any given $u$ on the domain under consideration, we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0; \tag{1.10}$$