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QUANTUM EULER-POISSON SYSTEM: LOCAL EXISTENCE OF SOLUTIONS

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Abstract The one-dimensional transient quantum Euler-Poisson system for semiconductors is studied in a bounded interval. The quantum correction can be interpreted as a dispersive regularization of the classical hydrodynamic equations and mechanical effects.

The existence and uniqueness of local-in-time solutions are proved with lower regularity and without the restriction on the smallness of velocity, where the pressure-density is general (can be non-convex or non-monotone).

Key Words Quantum Euler-Poisson system; existence of local classical solutions; non-linear fourth-order wave equation.

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1. Introduction

In 1927, Madelung gave a fluid-dynamical description of quantum systems governed by the Schrödinger equation for the wave function ψ:

$$\begin{aligned} i\varepsilon\partial_t\psi &= -\frac{\varepsilon^2}{2}\Delta\psi - V\psi \quad \text{in} \quad \mathbb{R}^d\times(0,\infty),\\ \psi(\cdot,0) &= \psi_0 \quad \text{in} \quad \mathbb{R}^d, \end{aligned}$$

where $d \geqslant 1$ is the space dimension, $\varepsilon > 0$ denotes the scaled Planck constant, and $V = V(x,t)$ is some (given) potential. By separating the amplitude and phase of $\psi = |\psi|\exp(iS/\varepsilon)$, the particle density $\rho = |\psi|^2$ and the particle current density $j = \rho\nabla S$ for irrotational flow satisfy the so-called *Madelung equations* [1]

$$\partial_t\rho + \operatorname{div} j = 0, \tag{1.1}$$

$$\partial_t j + \operatorname{div}\left(\frac{j\otimes j}{\rho}\right) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\rho}\right) = 0 \quad \text{in } \mathbb{R}^d\times(0,\infty). \tag{1.2}$$

The equations (1.1)-(1.2) can be interpreted as the pressureless Euler equations including the quantum Bohm potential

$$\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\rho}. \tag{1.3}$$

They have been used for the modelling of superfluids like Helium II [2, 3].

Recently, Madelung-type equations have been derived to model quantum phenomena in semiconductor devices, like resonant tunnelling diodes, starting from the Wigner-Boltzmann equation [4] or from a mixed-state Schrödinger-Poisson system [5, 6]. There are several advantages to the fluid-dynamical description of quantum semiconductors. First, kinetic equations, like the Wigner equation, or Schrödinger systems are computationally very expensive, whereas for Euler-type equations efficient numerical algorithms are available [7, 8]. Second, the macroscopic description allows for a coupling of classical and quantum models. Indeed, setting the Planck constant ε in (1.2) equal to zero, we obtain the classical pressureless equations. So in both pictures, the same (macroscopic) variables can be used. Finally, as semiconductor devices are modelled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or for the wave function.

The Madelung-type equations derived by Gardner [4] and Gasser et al. [5] also include a pressure term and a momentum relaxation term taking into account of interactions of the electrons with the semiconductor crystal, and are self-consistently coupled to the Poisson equation for the electrostatic potential ϕ:

$$\partial_t\rho + \mathrm{div} j = 0, \tag{1.4}$$

$$\partial_t j + \mathrm{div}\left(\frac{j\otimes j}{\rho}\right) + \nabla P(\rho) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\rho}\right) = -\frac{j}{\tau}, \tag{1.5}$$

$$\lambda^2\Delta\phi = \rho - \mathcal{C}(x) \quad \text{in } \Omega\times(0,\infty), \tag{1.6}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain, $\tau>0$ is the (scaled) momentum relaxation time constant, $\lambda>0$ the (scaled) Debye length, and $\mathcal{C}(x)$ is the doping profile modelling the semiconductor device under consideration [9, 10]. The pressure is assumed to depend only on the particle density and, like in classical fluid dynamics, often to have the expression

$$P(\rho) = \frac{T_0}{\gamma}\rho^\gamma, \quad \rho\geqslant 0, \quad \gamma\geqslant 1, \tag{1.7}$$

with the temperature constant $T_0>0$ employed [4, 11]. *Isothermal* fluids correspond to $\gamma=1$, *isentropic* fluids to $\gamma>1$. Notice that the particle temperature is $T(\rho)=T_0\rho^{\gamma-1}$.

The equations (1.4)-(1.6) are referred to as the *quantum Euler-Poisson system* or the *quantum hydrodynamic model*.

In this paper, we investigate the local existence of solutions of the following one-dimensional quantum Euler-Poisson problem:

$$\rho_t + (\rho u)_x = 0, \tag{1.8}$$