J. Partial Diff. Eqs. **16**(2003), 255–265
©International Academic Publishers Vol.16, No.3

MEAN CURVATURE FLOW OF GRAPHS IN $\Sigma_1 \times \Sigma_2$

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(Received Nov. 11, 2002)

Abstract Let Σ_1 and Σ_2 be m and n dimensional Riemannian manifolds of constant curvature respectively. We assume that w is a unit constant m-form in Σ_1 with respect to which Σ_0 is a graph. We set $v = \langle e_1 \wedge \cdots \wedge e_m, w\rangle$, where $\{e_1, \cdots, e_m\}$ is a normal frame on Σ_t. Suppose that Σ_0 has bounded curvature. If $v(x, 0) \geq v_0 > \frac{\sqrt{2}}{2}$ for all x, then the mean curvature flow has a global solution $\mathbf{F}$ under some suitable conditions on the curvatrue of Σ_1 and Σ_2.

Key Words Mean curvature flow; m-dimensional graphs.
2000 MR Subject Classification 53C44.
Chinese Library Classification O186.16.

1. Introduction

The mean curvature flow was first studied by Brakke from the viewpoint of geometric measure theory. Huisken investigated the smooth compact hypersurfaces moving by their the mean curvature. For higher co-dimensional case, Chen-Tian [1] and Chen-Li [2] studied mean curvature flow of surfaces in Kähler-Einstein surfaces, Chen-Li-Tian studied the mean curvature flow of two dimensional graphs in R^4, they proved in [3] that if there is a unit constant 2-form ω on R^4 such that $\langle e_1 \wedge e_2, \omega\rangle \geq v_0 > 0$ where e_1, e_2 is an orthonormal frame on Σ and if v_0 is larger than a fixed number on the initial surface, then the mean curvature flow has a global solution. Some similar results are also obtained by Wang [4].

In this paper, we generalize Chen-Li-Tian's global existence of the mean curvature flow to the case of m-dimensional graphs in $\Sigma_1 \times \Sigma_2$, where Σ_1 and Σ_2 are m and n dimensional Riemannian manifolds of constant curvature respectively ,$m, n \geq 2$. During the preparation of this paper, Jingyi Chen told us that wang [5] also got a longtime existence for the mean curvature flow. So we only consider the case that he did not study.

Let Σ be an m-dimensional oriented Riemannian manifold and let $\mathbf{F}_0 : \Sigma \to \Sigma_1 \times \Sigma_2$ be an immersion, Σ_1 and Σ_2 be m and n dimensional Riemannian manifolds of constant curvature k_1 and k_2 respectively. We denote $\Sigma_0 = \mathbf{F}_0(\Sigma)$. We say that Σ_0 is a graph if there exists an orthonormal frame $e_1, \cdots, e_m$ on Σ_0 and a unit constant m-form ω in Σ_1 such that

$$v = \langle e_1 \wedge \cdots \wedge e_m, \omega \rangle \geq v_0 > 0$$

for some constant v_0.

We consider a one-parameter family $\mathbf{F}_t = \mathbf{F}(\cdot, t)$ of submanifods with corresponding images $\Sigma_t = \mathbf{F}_t(\Sigma)$ such that

$$\begin{cases} \dfrac{d}{dt}\, \mathbf{F}(x,t) = \mathbf{H}(x,t) \\ \mathbf{F}(x,0) = \mathbf{F}_0(x) \end{cases} \tag{1.1}$$

where $\mathbf{H}(x,t)$ is the mean curvature vector of Σ_t at $\mathbf{F}(x,t)$. We assume that w is a unit constant m-form in Σ_1 with respect to which Σ_0 is a graph, we set $v = \langle e_1 \wedge \cdots \wedge e_m, w \rangle$, where $e_1, \cdots, e_m$ is a normal frame on Σ_t. Suppose that Σ_0 has bounded curvature. Our main result is that if $k_1 \geq k_2$ and $k_1 + k_2 > 0$, or $k_1 > 0$ and $k_1 + k_2 < 0$, where k_1 and k_2 are the constant curvature of Σ_1 and Σ_2. Assume that $v(x,0) \geq v_0 > \frac{\sqrt{2}}{2}$ for all x, then the equation (1.1) has a global solution $\mathbf{F}$.

Throughout this paper, the summation is taken for all repeated indices. We always assume that $m, n \geq 2$ and adopt the following ranges of indices

$$\begin{aligned} 1 \leq\ i,\ j,\ \cdots\ &\leq m, \\ m+1 \leq \alpha,\ \beta,\ \gamma,\ \cdots\ &\leq m+n. \end{aligned}$$

The authors would like to thank Xiaoli Han for his helpful discussions.

2. Preliminaries Results

We assume that $\mathbf{F}(x,t)$ satisfies the mean curvature flow equation (1.1). Suppose that $\mathbf{H}$ is the mean curvature vector of the Riemannian manifold $\mathbf{F}(\Sigma, t)$ in $\Sigma_1 \times \Sigma_2$, $\mathbf{A}$ is the second fundamental form and denote the Riemannian metric on $\Sigma_1 \times \Sigma_2$ by $\langle \cdot, \cdot \rangle$. In a normal coordinates around a point in Σ, the induced metric on Σ_t from $\langle \cdot, \cdot \rangle$ is given by $g_{ij} = \langle \partial_i \mathbf{F}, \partial_j \mathbf{F} \rangle$ where ∂_i $(i = 1, \cdots, m)$ are the partial derivatives with respect to the local coordinates. Let $\triangle$ and ∇ be the Laplace operator and the covariant derivative for the induced metric on Σ_t respectively. We choose an orthonormal frame $e_1, \cdots, e_m, v_1, \cdots, v_n$ of $\Sigma_1 \times \Sigma_2$ such that $e_1, \cdots, e_m$ is a frame of $\Sigma_t = \mathbf{F}(\Sigma, t)$, and $v_1, \cdots, v_n$ is a frame of the normal bundle over Σ_t. We can write:

$$\begin{aligned} \mathbf{A} &= A^\alpha v_\alpha, \\ \mathbf{H} &= -H^\alpha v_\alpha. \end{aligned}$$