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SOME REMARKS ABOUT HEISENBERG GROUPS*

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Abstract In this paper, we give two inequalities and another characterizations about Heisenberg groups which do not coincide with the related results in classical case.

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1. Introduction

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Let $\mathbf{H}^m$ denote Heisenberg group which is a Lie group that has algebra $\mathbf{g} = R^{2m+1}$, with a nonablian group law:

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = \Big(x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(y_2 x_1 - x_2 y_1)\Big), \tag{1.1}$$

for every $u_1 = (x_1, y_1, t_1)$, $u_2 = (x_2, y_2, t_2) \in \mathbf{H}^m$. The Lie algebra is generated by the left invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, \cdots, m, \tag{1.2}$$

and $T = \frac{\partial}{\partial t}$. For every $u_1 = (x_1, y_1, t_1)$, $u_2 = (x_2, y_2, t_2) \in \mathbf{H}^m$, the metric $d(u_1, u_2)$ in the Heisenberg group $\mathbf{H}^m$ is defined as([1])

$$\begin{aligned} d(u_1, u_2) &= |u_2 u_1^{-1}| \\ &= \Big[\Big((x_2 - x_1)^2 + (y_2 - y_1)^2\Big)^2 + \Big(t_2 - t_1 + 2(x_2 y_1 - x_1 y_2)\Big)^2\Big]^{\frac{1}{4}}. \end{aligned} \tag{1.3}$$

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In particular, for $u=(x,y,t)=(z,t)\in \mathbf{H}^m$ and $z=x+iy$ with dilation $|(sz,s^2t)|=s|(z,t)|$ for $s>0$, a homogeneous gauge $|u|$ is defined as $\left[(x^2+y^2)^2+t^2\right]^{\frac14}=(|z|^4+t^2)^{\frac14}$.

We may see that the $\mathbf{H}^m$ possesses the nonlinear structure of the group law which is one of the differences between $\mathbf{H}^m$ and general Riemann manifold. Indeed, the geometry of $\mathbf{H}^m$ is not Euclidean at every scale since it was proved by S. Semmes([2]). Recently, the fact that $\mathbf{H}^m$ is such a singular space that can be intuitively understood also in the light of a recent result of Christodoulou [3] who proved that the Heisenberg group can be constructed as the continuum limit of a crystalline material.

In this paper, we introduce another properties about Heisenberg groups where some results do not coincide with the results in classical case.

2. Some Inequalities about Heisenberg Groups

In this section, we study some type inequalities about Heisenberg groups, which may take important part in obtaining some analytic properties.

Lemma 2.1($\mathbf{C}_p$-inequality) *Let $p>0$, for any $a_i\in R$, then*

$$\left(\sum_{i=1}^{n}|a_i|\right)^p\le C_p\sum_{i=1}^{n}|a_i|^p,$$

where $C_p=1$ if $0<p<1$ and $C_p=n^{p-1}$ if $p\ge 1$.

Theorem 2.2 *If $u_1=(x_1,y_1,t_1)=(z_1,t_1), u_2=(x_2,y_2,t_2)=(z_2,t_2)\in \mathbf{H}^m$, and $z_k=x_k+iy_k$, $x_k,y_k\in R^m$, $k=1,2$, then*

$$|u_1^{-1}\cdot u_2|\le |u_1|+|u_2|. \tag{2.1}$$

About distance function $d(u,v)$, S.Semmes pointed out that it satisfies the quasi-triangle inequality $d(u,v)\le K(d(u,w)+d(v,w))$ for some constant $K>0$ and for any $u,v,w\in \mathbf{H}^m$ in [4]. In the following theorem, we will give a more precise quasi-triangle inequality.

Theorem 2.3 *If $u=(x_u,y_u,t_u), v=(x_v,y_v,t_v), w=(x,y,t)\in \mathbf{H}^m$, where x_u, x_v, x, y_u, y_v, $y\in R^m$, t_u, t_v, $t\in R$, then*

$$d(u,v)\le 62^{\frac14}\Big(d(u,w)+d(v,w)\Big). \tag{2.2}$$

The proofs about Theorem 2.2 and Theorem 2.3 are quite elementary and the details are omitted here.

we would conjecture that there are $u,v,w\in \mathbf{H}^m$, such that $d(u,v)\ge d(u,w)+d(v,w)$. But we cannot prove this at present.