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STRUCTURE OF POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH CRITICAL AND SUPERCRITICAL GROWTH

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Abstract The structure of positive radial solutions to a class of quasilinear elliptic equations with critical and supercritical growth is precisely studied. A large solution and a small solution are obtained for the equations. It is shown that the large solution is unique, its asymptotic behaviour and flat core are also discussed.
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1. Introduction

In this paper we consider the quasilinear elliptic problem

$$-\Delta_p u = u^k - \epsilon u^q \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \tag{I_ϵ}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$; $q > k \geq k_N := \dfrac{(N+1)p - N}{N-p}$; $\epsilon > 0$; B is the unit ball in $\mathbf{R}^N$ with $N > p$. Clearly, (I_ϵ) is a purterbed problem of the problem

$$-\Delta_p u = u^k \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B. \tag{1.1}$$

The Pohozaev's identity (see [1]) implies that (1.1) has no positive solution.

We are interested in the structure of positive radial solutions of (I_ϵ) when ϵ is sufficiently small. By a positive solution u_ϵ of (I_ϵ), we mean that $u_\epsilon \in W_0^{1,p}(B) \cap C^1(\overline{B})$, $u_\epsilon > 0$ in B and

$$\int_B |Du_\epsilon|^{p-2} Du_\epsilon \cdot D\varphi dx = \int_B \left[u_\epsilon^k - \epsilon u_\epsilon^q\right] \varphi dx$$

for all $\varphi \in W_0^{1,p}(B)$.

We show that the problem (I_ϵ), which we call the *approach problem*, has at least two positive radial solutions: $\overline{u}_\epsilon$, $\underline{u}_\epsilon$ for $\epsilon > 0$ sufficiently small; $\overline{u}_\epsilon$ is a large solution and $\underline{u}_\epsilon$ is a small solution. By a large positive radial solution u_ϵ of (I_ϵ), we mean that there exists $0 < r_0 < 1$ (independent of ϵ) such that

$$\liminf_{\epsilon\to 0} \epsilon^{1/(q-k)} u_\epsilon(r) > 0 \ \text{ for } \ r \in [0, r_0]. \tag{1.2}$$

By a small positive radial solution u_ϵ of (I_ϵ), we mean that u_ϵ is a positive solution of (I_ϵ) and $\epsilon^{1/(q-k)}u_\epsilon \to 0$ in any compact set in $[0,1]\backslash\{0\}$ as $\epsilon \to 0$. Moreover, we also show that the large positive radial solution of (I_ϵ) is unique when ϵ is sufficiently small.

Now, we write the equation in (I_ϵ) to the form:

$$-\Delta_p(\epsilon^{1/(q-k)}u) = \epsilon^{-(k-p+1)/(q-k)}\Big[(\epsilon^{1/(q-k)}u)^k - (\epsilon^{1/(q-k)}u)^q\Big].$$

Setting $w = \epsilon^{1/(q-k)}u$, $\lambda = \epsilon^{-(k-p+1)/(q-k)}$ and $\xi(s) = s^k(1 - s^{q-k})$, we obtain an equivalent form of (I_ϵ) :

$$-\Delta_p w = \lambda\xi(w) \ \text{ in } \ B, \quad w = 0 \ \text{ on } \ \partial B. \tag{J_λ}$$

Moreover, $\lambda \to +\infty$ as $\epsilon \to 0^+$ since $k + 1 \geq \frac{Np}{N-p} > p$. We shall always use the form (J_λ) in the follows.

The problem (J_λ) with $p > 2$ has been studied by many authors in general smooth domains Ω. There are a few works on the *equidiffusive case* $p = k + 1$ as follows. Let λ_1 be the first eigenvalue of $-\Delta_p$ under zero Dirichlet boundary condition. In the one-dimensional case $N = 1$, Guedda and Véron [2] have shown by phase-plane analysis that if $\lambda > \lambda_1$, then (J_λ) has a unique positive solution w_λ with $\|w_\lambda\|_\infty \leq 1$, and that a set called *the flat core of* w_λ,

$$\mathcal{O}_\lambda = \mathcal{O}_\lambda(w_\lambda) := \{x \in \Omega : \ w_\lambda(x) = 1\} \tag{1.3}$$

is non-empty for sufficiently large λ. Since the length of $\mathcal{O}_\lambda$ can be indicated explicitly, we can see that as $\lambda \to \infty$, $\mathcal{O}_\lambda$ spreads out toward the whole of Ω with the growth as

$$\lim_{\lambda\to\infty} \lambda^{1/p}\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) = C(\xi, p) \tag{1.4}$$

where $C(\xi,p) = \Big(\dfrac{p-1}{p}\Big)^{1/p}\displaystyle\int_0^1 (\Xi(1) - \Xi(s))^{-1/p}ds$ and $\Xi(s) = \displaystyle\int_0^s \xi(t)dt$. In the higher-dimensional case $N \geq 2$, the phase-plane analysis is no longer useful and one has to use other methods. Constructing a suitable subsolution by using eigenfunction for λ_1, Kamin and Véron [3] have proved that the unique solution of (J_λ) has a flat core for sufficiently large λ and extended the results of [2]. However, they gave only an estimate $\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C\lambda^{-1/p}$ as $\lambda \to \infty$, where C is a constant independent of λ, without explicit information about C and any estimate of $\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega)$ from below. In virtue of an exact estimate for $\mathcal{O}_\lambda$, Melián and de Lis [4] have utilized the solutions for $N = 1$, whose dependence on λ is understood well, to make super- and subsolutions and conclude that (1.4) also holds true in the case $N \geq 2$. Under a stronger condition