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THE PERIODIC INITIAL VALUE PROBLEM AND INITIAL VALUE PROBLEM FOR THE MODIFIED BOUSSINESQ APPROXIMATION

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Abstract The Boussinesq approximation, where the viscosity depends polynomially on the shear rate,finds more and more frequent use in geological practice. In this paper, we consider the periodic initial value problem and initial value problem for this modified Boussinesq approximation with the viscous part of the stress tensor $\tau^v = \tau(\mathbf{e}) - 2\mu_1\Delta\mathbf{e},$ where the nonlinear function $\tau(\mathbf{e})$ satisfies $\tau_{ij}(\mathbf{e})e_{ij} \geq C\,|\mathbf{e}|^p$or $\tau_{ij}(\mathbf{e})e_{ij} \geq C(|\mathbf{e}|^2 + |\mathbf{e}|^p)$. The existence, uniqueness and regularity of the weak solution is proved for $p > \dfrac{2n}{n+2}$.

Key Words non-Newtonian incompressible fluids, Boussinesq approximation, periodic initial value problem, initial value problem,weak solution

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1. Introduction

The study of flows in the Earth's mantle consists of thermal convection in a highly viscous fluid. For a description of dynamics of flows of an incompressible fluid in processes where the thermal effects play an essential role,the Boussinesq approximation is a reasonable model to present essential phenomena of such flows and it is used, for example, in planetary physics for describing processes in body interior[1,2].

Let $n = 2$ *or* 3. we denote $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ the velocity field associated with the flow of an incompressible bipolar viscous fluid , π the pressure and θ the temperature. The Boussinesq approximation in nondimensional form is described by

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\pi + div\tau^v + \rho\mathbf{e}_n\theta + \mathbf{f}. \tag{1.1}$$

$$\mathrm{div}\mathbf{u} = \frac{\partial u_i}{\partial x_i} = 0, \tag{1.2}$$

$$\rho\frac{\partial \theta}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\theta - \kappa\Delta\theta = g, \tag{1.3}$$

where $\rho = const > 0$ is the density, $\kappa > 0$ is a positive constant coefficient (thermometric conductivity), $\mathbf{e}_n$ is a unit vector in R^n, and $\mathbf{f}(x,t)$, $g(x,t)$ are given vector value and scalar functions, respectively. For the sake of simplicity, we put $\rho = 1$ and $\kappa = 1$. We refer to Padula [3] and Hills and Roberts [4] for a derivation of the Boussinesq approximation (1.1)—(1.3).

In order to make the system of equations complete it is necessary to prescribe the constitutive relation for the viscous part of the stress tensor . In the present work we will assume that the constitutive laws have the form

$$\tau^v = \tau(\mathbf{e}) - 2\mu_1 \triangle \mathbf{e}, \tag{1.4}$$

where $\mu_1 > 0$ and the $\tau(\mathbf{e})$ is given via a scalar potential U of the symmetrized velocity gradient $\mathbf{e}$, i.e.

$$\tau_{ij}(\mathbf{e}) = \frac{\partial U(\mathbf{e})}{\partial e_{ij}}, \qquad i,j = 1,2,\cdots,n. \tag{1.5}$$

$$\mathbf{e}(\mathbf{u}) = (e_{ij}(\mathbf{u})), \quad e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.6}$$

$U(\cdot)$ is twice continuously differentiable in R^n,$U \geq 0, U(0) = \dfrac{\partial U(0)}{\partial e_{ij}} = 0$ for all $i,j = 1$, 2, $\cdots, n$ .and such that

$$\frac{\partial^2 U(\mathbf{e})}{\partial e_{ij}\partial e_{kl}}\xi_{ij}\xi_{kl} \geq c_1 \begin{cases} |\mathbf{e}|^{p-2}|\xi|^2, & p < 2, \\ (1+|\mathbf{e}(\mathbf{u})|)^{p-2}|\xi|^2, & p \geq 2, \end{cases} \tag{1.7}$$

$$\left|\frac{\partial^2 U(\mathbf{e})}{\partial e_{ij}\partial e_{kl}}\right| \leq c_2(1+|\mathbf{e}|)^{p-2}, \tag{1.8}$$

where $p \in (1,\infty), c_1, c_2 > 0, \xi \in R^n_{sym} \equiv \{M \in R^{n^2}; M_{ij} = M_{ji}, i,j = 1,2,...,n\}, |\mathbf{e}| = (e_{ij}e_{ij})^{1/2}$.From the hypothesis of U above we can shown that

$$|\tau_{ij}(\mathbf{e})| \leq c_3(1+|\mathbf{e}(\mathbf{u})|)^{p-1}, p \geq 2, \tag{1.9}$$

$$|\tau_{ij}(\mathbf{e})| \leq c_3 |\mathbf{e}(\mathbf{u})|^{p-1}, 1 \leq p < 2, \tag{1.10}$$

as well as

$$\tau_{ij}(\mathbf{e})e_{ij} \geq c_4 |\mathbf{e}(\mathbf{u})|^p, \qquad 1 < p < \infty, \tag{1.11}$$

we note that for the following nonlinear models

$$\tau_{ij}(\mathbf{e}) = 2\mu_0(\varepsilon + |\mathbf{e}(\mathbf{u})|^{p-2})e_{ij}, \qquad p > 2, \ \varepsilon > 0, \tag{1.12}$$

and

$$\tau_{ij}(\mathbf{e}) = 2\mu_0 |\mathbf{e}(\mathbf{u})|^{p-2} e_{ij}, \qquad 1 < p < 2, \tag{1.13}$$