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PERSISTENT HOMOCLINIC ORBITS FOR A PERTURBED CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract In this paper, the existence of homoclinic orbits, for a perturbed cubic-quintic nonlinear Schrödinger equation with even periodic boundary conditions, under the generalized parameters conditions is established. More specifically, we combine geometric singular perturbation theory with Melnikov analysis and integrable theory to prove the persistence of homoclinic orbits.

Key Words Homoclinic orbit; perturbed cubic-quintic nonlinear Schrödinger equation; geometric singular perturbation theory; Melnikov analysis.

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1. Introduction

In recent years, there have been extensive studies on the existence of homoclinic orbits for near integrable dissipative PDEs, which are closely related to chaos. In this work, we consider a perturbed cubic-quintic nonlinear Schrödinger equation (CQS)

$$iq_t = q_{xx} + 2(q\,\bar{q} - \omega^2)q + i\varepsilon(\hat{D}q - \Gamma - m\,|q|^4\,q - n\,|q|^2\,q), \tag{1.1}$$

where q is 2π periodic and even in x, $\hat{D}$ is a bounded dissipative operator and is assumed to take the form

$$\hat{D}q = -\alpha q + \beta\hat{B}q \tag{1.2}$$

for positive constants α and β. Here $\hat{B}$ is a Fourier truncation of ∂_{xx}, i. e.

$$\hat{B}\cos(kx) = \begin{cases} -k^2\cos(kx)\,, & k < K, \\ 0\,, & k \geq K, \end{cases} \tag{1.3}$$

the constants ω, m, and n are assumed to satisfy $\omega \in (\frac{1}{2}, 1), m \geq 0$, and $n \geq 0$; and $\varepsilon > 0$ is a small perturbation parameter. We shall prove that, for sufficiently small $\varepsilon > 0$

and appropriate parameters, there exists a solution of the equation (1.1) homoclinic to an equilibrium.

When $m = n = 0$, the perturbed nonlinear Schrödinger equation (NLS)

$$iq_t = q_{xx} + 2(q\,\bar{q} - \omega^2)q + i\varepsilon(\hat{D}q - \Gamma) \tag{1.4}$$

is studied numerically, one finds solutions that, at large time $t \gg 1$, consist of very regular spatial patterns that oscillate irregularly, maybe even chaotically, in time t. Details can be found in [1]. This chaostic behavior is believed to be closely related to persistent homoclinic structures from the original 'figure 8' structures of integrable systems, composed of whiskered tori and their homoclinic connections. Thus, the proof of the existence of homoclinic orbits for the perturbed NLS is naturally the next step.

The analysis of the existence of homoclinic orbits for the perturbed NLS was initially carried out for finite dimensional versions, first, a four-dimensional Fourier truncation, in [2–5], and then a finite $(2N + 2)$ −dimensional finite difference discretization, in [6] and [7]. In[8], the existence of homoclinic orbits was proved for the equation (1.4) at $\Gamma = 1$. When the bounded operator $\hat{B}$ is replaced by an unbounded operator ∂_{xx}, [9] proved the existence of homoclinic orbits for the perturbed NLS equation.

When $\varepsilon = 0$,the unperturbed CQS is a completely integrable NLS equation. It has a temporally periodic solution

$$q\,(t) = r\exp\{-i\left[2\left(r^2 - \omega^2\right)t - \theta\right]\},$$

where $r \neq \omega$ and θ denotes real constants.

Using Bäcklund-Darboux transformation from soliton mathematics, one can obtain an exact solutions of NLS

$$q_h^{\pm}\,(t) = \left\{\frac{\cos 2p\cosh\tau - i\sin 2p\sinh\tau \pm \sin p\cos x}{\cosh\tau \mp \sin p\cos x}\right\}q, \tag{1.5}$$

where $\tau = \sigma_r\,(t + t_0)\,, \sigma_r = \sqrt{4r^2 - 1}$, and $p = \tan^{-1}\sigma_r$.

It is easy to see that $q_h^{\pm}\,(t)$ are homoclinic to $q\,(t)$ with a phase shift $-4p$, when $r = \omega, C_\omega = \{q \mid q_x = 0, |q| = \omega\}$ is a circle of fixed points and $q_h^{\pm}\,(t)$ are actually heteroclinic orbits. We shall start with these homoclinic and heteroclinic orbits to construct homoclinic orbits for the perturbed CQS. More specifically, when $\varepsilon > 0$, for the equation (1.1), the circle C_ω of fixed points breaks and a saddle Q appears in a neighborhood of C_ω. We shall prove that, for sufficiently small $\varepsilon > 0$, there exists a solution homoclinic to Q.

Note that the equation (1.1) is in the form of the important complex Ginzburg-Landau equations. Recently, one also studied Ginzburg-Landau equations as perturbation of NLS which preserve phase symmetry. In a series [10–12], they discussed the persistence of special solutions such as rotating waves, travelling waves, and quasi-periodic solutions through Melnikov approach. Some necessary conditions were derived by utilizing some invariants of NLS.