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GLOBAL APPROXIMATELY CONTROLLABILITY AND FINITE DIMENSIONAL EXACT CONTROLLABILITY FOR PARABOLIC EQUATION*

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Abstract We study the globally approximate controllability and finite-dimensional exact controllability of parabolic equation where the control acts on a mobile subset of Ω ,or, a curve in $Q = \Omega \times (0, T)$.
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1. Introduction

Let Ω be a bounded, open, connected set in R^n with boundary $\partial\Omega$. Consider the following homogeneous Dirichlet problem for the parabolic equation:

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - a(x,t)u$$
$$\text{in} \quad Q = (0,T) \times \Omega, \tag{1.1a}$$
$$u|_\Sigma = 0 \quad \text{in} \quad \Sigma = \partial\Omega \times (0,T), \ \ u|_{t=0} = u_0 \quad \text{in} \quad \Omega,$$

under the condition of uniform ellipticity, namely,

$$\mu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ \ \forall \xi_i \in R \quad \text{a.e.} \quad \text{in} \quad Q, \ \mu > 0, \tag{1.1b}$$

where $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(Q)$, $i, j = 1, ..., n$. To guarantee the solvability and unique continuation, some other assumptions are needed [1-2]:

$$u_0 \in L^2(\Omega), \ \| \sum_{i=1}^{n} b_i^2, a\|_{q,r,Q} \le \mu, \quad \frac{1}{r} + \frac{n}{2q} = 1 - k, \tag{1.2a}$$

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$$\begin{cases} q \in [\frac{n}{2(1-k)}, \infty],\ r \in [\frac{1}{1-k}, \infty],\ 0 < k < 1, \quad \text{for} \quad n \geq 2, \\ q \in [1, \infty],\ r \in [\frac{1}{1-k}, \frac{2}{1-2k}],\ 0 < k < \frac{1}{2}, \quad \text{for} \quad n = 1, \end{cases} \tag{1.2b}$$

where $\|z\|_{q,r,Q} = (\int_0^T (\int_\Omega |z|^q dx)^{\frac{r}{q}} dt)^{\frac{1}{r}}$;

$$\frac{\partial a_{ij}}{\partial t} \in L^1(0,T;L^\infty(\Omega)),\ b_i, a \in L^\infty(Q), \tag{1.3}$$

$$u_0 \in H_0^1(\Omega),\ \partial\Omega \in C^2,\ \frac{\partial a_{ij}}{\partial x_k}, b_i, a \in L^\infty(Q). \tag{1.4}$$

The conditions (1.2) ensure the existence and uniqueness of a solution to (1.1) from the space $C([0,T];L^2(\Omega)) \cap H_0^{1,0}(Q)$(see Ladyzenskaja [1]), which satisfies the energy estimate:

$$\|u\|_{C([0,T];L^2(\Omega))} + \|u\|_{H^{1,0}(Q)} \leq c\|u_0\|_{L^2(\Omega)}. \tag{1.5}$$

Here c depends on T and the parameters in (1.1b), (1.2). Under the assumptions (1.4) this solution lies in $H_0^{2,1}(Q)$. The assumptions (1.3) allow one to use the backward uniqueness result.

The reference [2] gives the following unique continuation results:

Proposition 1.1 *Let $n \leq 3$. Given $T > \epsilon > 0$, there exists a measurable curve $(\epsilon, T) \ni t \to \hat{x}(t) \in \bar{\Omega}$ such that every solution $u \in H_0^{2,1}(Q)$ to (1.1), (1.3), (1.4) which vanishes along $\hat{x}(\cdot)$ and vanishes in Q.*

Proposition 1.2 *Given $T > \epsilon > 0$, there exists a set-valued map $(\epsilon, T) \ni t \to S(t) \subset \Omega$, $\text{mes}\{S(t)\} > 0$ such that every solution $u \in C([0,T];L^2(\Omega)) \cap H_0^{1,0}(Q)$ to (1.1), (1.2), (1.3) which satisfies the equality $\int_{S(t)} udx = 0$ on (ϵ, T) vanishes in Q.*

Furthermore, [2] studies the approximate controllability of the following control system:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} =& \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x, T-t)\frac{\partial \varphi}{\partial x_j}) \\ &+ \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i(x,T-t)\varphi) - a(x,T-t)\varphi + B(T-t)v(t) \quad \text{in} \quad Q, \\ \varphi =& 0 \quad \text{in}\, \Sigma,\ \ \varphi|_{t=0} = 0, \end{aligned} \tag{1.6}$$

where $B(\cdot)$ is a linear operator defined on a linear manifold $V \subseteq L^2(0,T)$ by one of the following formulas:

$$B(T-t)v(t) = v(t) \times \begin{cases} 1, \text{ if } & x \in S(T-t), \\ 0, \text{ if } & x \notin S(T-t), \end{cases} \quad S(t) \subset \Omega \quad \text{a.e. in} \quad [0,T], \tag{1.7}$$

or

$$B(T-t)v(t) = v(t)\delta(x - \hat{x}(T-t)), \quad \hat{x}(t) \in \bar{\Omega} \quad \text{a.e. in} \quad [0,T], \tag{1.8}$$