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On the Change of Variables Formula for Multiple Integrals

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Abstract. We develop an elementary proof of the change of variables formula in multiple integrals. Our proof is based on an induction argument. Assuming the formula for $(m-1)$-integrals, we define the integral over hypersurface in $\mathbb{R}^m$, establish the divergent theorem and then use the divergent theorem to prove the formula for m-integrals. In addition to its simplicity, an advantage of our approach is that it yields the Brouwer Fixed Point Theorem as a corollary.

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1 Introduction

The change of variables formula for multiple integrals is a fundamental theorem in multivariable calculus. It can be stated as follows.

Theorem 1.1. *Let D and Ω be bounded* open *domains in $\mathbb{R}^m$ with piece-wise C^1-boundaries, $\varphi \in C^1(\bar{\Omega},\mathbb{R}^m)$ such that $\varphi:\Omega\to D$ is a C^1-diffeomorphism. If $f\in C(\bar{D})$, then*

$$\int_D f(y)\mathrm{d}y=\int_\Omega f(\varphi(x))\left|J_\varphi(x)\right|\mathrm{d}x, \tag{1.1}$$

where $J_\varphi(x)=\det\varphi'(x)$ is the Jacobian determinant of φ at $x\in\Omega$.

The usual proofs of this theorem that one finds in advanced calculus textbooks involves careful estimates of volumes of images of small cubes under the map φ and numerous annoying details. Therefore several alternative proofs have appeared in recent years. For example, in [5] P. Lax proved the following version of the formula.

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Theorem 1.2. *Let $\varphi:\mathbb{R}^m\to\mathbb{R}^m$ be a C^1-map such that $\varphi(x)=x$ for $|x|\geq R$, and $f\in C_0(\mathbb{R}^m)$. Then*

$$\int_{\mathbb{R}^m} f(y)\mathrm{d}y=\int_{\mathbb{R}^m} f(\varphi(x))J_\varphi(x)\mathrm{d}x.$$

The requirment that φ is an identity map outside a big ball is somewhat restricted. This restriction was also removed by Lax in [6]. Then, Tayor [7] and Ivanov [4] presented different proofs of the above result of Lax [5] using differential forms. See also Báez-Duarte [1] for a proof of a variant of Theorem 1.1 which does not require that $\varphi:\Omega\to D$ is a diffeomorphism. As pointed out by Taylor [7, Page 380], because the proof relies on integration of differential forms over manifolds and Stokes' theorem, it requires that one knows the change of variables formula as formulated in our Theorem 1.1.

In this paper, we will present a simple elementary proof of Theorem 1.1. Our approach does not involve the language of differential forms. The idea is motivated by Excerise 15 of §1-7 in the famous textbook on classical differential geometry [3] by do Carmo. The excerise deals with the two dimensional case $m=2$. We will perform an induction argument to generize the result to the higher dimensional case $m\geq 2$. In our argument, we will apply the Cauchy-Binet formula about the determinant of the product of two matrics. As a byproduct of our approach, we will also obtain the Non-Retraction Lemma (see Corollary 3.2), which implies the Brouwer Fixed Point Theorem.

2 Integral over hypersurface

We will prove Theorem 1.1 by an induction argument. The case $m=1$ is easily proved in single variable calculus. Suppose we have proven Theorem 1.1 for $(m-1)$-dimension, where $m\geq 2$. We will define the integral over a hypersurface (of codimension one) in $\mathbb{R}^m$ and establish the divergent theorem in $\mathbb{R}^m$. Then, in the next two sections we will use the divergent theorem to prove Theorem 1.1 for m-dimension.

Let U be a Jordan measurable bounded closed domain in $\mathbb{R}^{m-1}$, $x:U\to\mathbb{R}^m$,

$$(u^1,\dots,u^{m-1})\mapsto(x^1,\dots,x^m)$$

be a C^1-map such that the restriction of x in the interior U° is injective, and

$$\operatorname{rank}\left(\frac{\partial x^i}{\partial u^j}\right)=m-1, \tag{2.1}$$

then we say that $x:U\to\mathbb{R}^m$ is a C^1-parametrized surface. By definition, two C^1-parametrized surfaces $x:U\to\mathbb{R}^m$ and $\tilde{x}:\tilde{U}\to\mathbb{R}^m$ are equivalent if there is a C^1-diffeomorphism $\phi:\tilde{U}\to U$ such that $\tilde{x}=x\circ\phi$. The equivalent class $[x]$ is called a *hypersurface*, and $x:U\to\mathbb{R}^m$ is called a parametrization of the hypersurface. Since it is easy to see that $x(U)=\tilde{x}(\tilde{U})$ if x and $\tilde{x}$ are equivalent, $[x]$ can be identified as the subset $S=x(U)$.