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Overdetermined Boundary Value Problems in $\mathbb{S}^n$

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Abstract. In this paper we use the maximum principle and the Hopf lemma to prove symmetry results to some overdetermined boundary value problems for domains in the hemisphere or star-shaped domains in $\mathbb{S}^n$. Our method is based on finding suitable P-functions as Weinberger ([26]).

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1 Introduction

In a seminal paper [21], Serrin proved that for a bounded open connected domain $\Omega \subset \mathbb{R}^n$ with sufficient regular boundary $\partial\Omega$, if there exists a solution of the following overdetermined boundary value problem

$$\begin{cases} \Delta u = n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where c is a constant, then Ω must be a ball and u is radially symmetric. Here ν denotes the outward unit normal of $\partial\Omega$.

The main tool of Serrin's proof is well-known as the method of moving planes, which is due to Alexandrov. Immediately after Serrin's paper, Weinberger [26] give an alternative proof of the same result, based on a Rellich-Pohozaev type identity and an interior maximum principle for a subharmonic function (In literatures, it is often referred to as P-function). Each of their proofs has its own merits. Serrin's argument applies to very

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general partial differential equations if an additional assumption $u>0$ is added, while Weinberger's argument is more elementary.

Since the works of Serrin and Weinberger, there have been numerous generalizations to overdetermined problems for general elliptic operators in $\mathbb{R}^n$, the interested readers may refer to [4–6, 8, 11–14, 17, 25] and references therein.

On the other hand, Serrin's result has been extended to the hemisphere $\mathbb{S}^n_+$ and the hyperbolic space $\mathbb{H}^n$. Precisely, Molzon [16] considered equation $\Delta u = f(x)$ where $f(x) = \cos r$ ($\cosh r$ resp.) in the case $\mathbb{S}^n_+$ ($\mathbb{H}^n$ resp.) and r is the distance function from a fixed point or $f(x)=n$. Kumaresan and Prajapat [15] considered equation $\Delta u + f(u) = 0$ in $\Omega \subset \mathbb{S}^n_+$ or $\mathbb{H}^n$, where f is a C^1 function. They proved that if $\Delta u + f(u) = 0$ with the boundary condition $u=0$ and $\frac{\partial u}{\partial \nu} = constant$ admits a *positive* solution, then Ω is a geodesic ball and u is radially symmetric. They used Serrin's method of moving planes to achieve this, where the positivity of u is an unremovable assumption.

In this paper, we will study an overdetermined problem corresponding to a particular equation on $\mathbb{S}^n$:

$$\begin{cases} \Delta u + nu = n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) is related to Schiffer's problem (See [28] problem 80) on $\mathbb{S}^n$. See for instance [1–3, 7, 9, 10, 23, 24, 27] for recent developments of Schiffer's problem. Previously, Souam [24] showed that for $n=2$ and $\Omega \subset \mathbb{S}^n$ *simply connected*, if (1.2) admits a solution, then Ω must be a geodesic ball.

Our first result is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{S}^n$ be a bounded open connected domain such that $\overline{\Omega}$ is contained in a hemisphere $\mathbb{S}^n_+$. If the overdetermined problem* (1.2) *admits a solution u, then Ω must be a geodesic ball and u is radially symmetric.*

We remark that since the first Dirichlet eigenvalue for a domain $\overline{\Omega} \subset \mathbb{S}^n_+$ is strictly larger than n, there exists a unique solution for the Dirichlet problem $\Delta u + nu = n$ in Ω and $u=0$ on $\partial\Omega$. However, it is not a priori known whether the solution has a definite sign. Therefore, Theorem 1.1 does not follow from the result of Kumaresan and Prajapat [15].

Our approach to Theorem 1.1 is parellel to Weinberger's, namely, we use a maximum principle for a subharmonic function P and a Rellich-Pohozaev type identity. We remark that our method also applies to equation $\Delta u - nu = n$ in $\Omega \subset \mathbb{H}^n$. In this case, u is negative in Ω by the maximum principle. Hence the conclusion also follows from the result of Kumaresan and Prajapat.

Our next result concerns the same overdetermined problem (1.2) in $\Omega \subset \mathbb{S}^n$ without the assumption that $\overline{\Omega}$ is contained in a hemisphere $\mathbb{S}^n_+$. Instead, we shall add a star-shapedness assumption on Ω. A domain $\Omega \subset \mathbb{S}^n$ is called star-shaped with respect to $p \in \mathbb{S}^n$ if Ω can be written as a graph over a geodesic sphere centered at p. It is clear that a