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# Optimal Parameters for Doubling Algorithms

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**Abstract.** In using the structure-preserving algorithm (SDA) [*Linear Algebra Appl.*, 2005, vol.396, pp.55–80] to solve a continuous-time algebraic Riccati equation, a parameter-dependent linear fractional transformation $z \to (z-\gamma)/(z+\gamma)$ is first performed in order to bring all the eigenvalues of the associated Hamiltonian matrix in the open left half-plane into the open unit disk. The closer the eigenvalues are brought to the origin by the transformation via judiciously selected parameter $\gamma$, the faster the convergence of the doubling iteration will be later on. As the first goal of this paper, we consider several common regions that contain the eigenvalues of interest and derive the best $\gamma$ so that the images of the regions under the transform are closest to the origin. For our second goal, we investigate the same problem arising in solving an $M$-matrix algebraic Riccati equation by the alternating-directional doubling algorithm (ADDA) [*SIAM J. Matrix Anal. Appl.*, 2012, vol.33, pp.170–194] which uses the product of two linear fractional transformations $(z_1,z_2) \to [(z_2-\gamma_2)/(z_2+\gamma_1)][(z_1-\gamma_1)/(z_1+\gamma_2)]$ that involves two parameters. Illustrative examples are presented to demonstrate the efficiency of our parameter selection strategies.

**AMS subject classifications**: 15A24, 65F30, 65H10

**Key words**: Doubling algorithm, SDA, ADDA, optimal parameter, algebraic Riccati equation.

## 1 Introduction

The following *so-called* continuous-time algebraic Riccati equation (CARE)

$$A^{\mathrm{T}}X + XA - XGX + H = 0 \tag{1.1}$$

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frequently arises from the continuous-time Linear-Quadratic Gaussian control (LQG) – the $\mathcal{H}_2$-control [12], where $A, G^{\mathrm{T}} = G, H^{\mathrm{T}} = H \in \mathbb{R}^{n\times n}$. Here and in what follows, the superscript T takes the matrix transpose and $\mathbb{R}^{n\times n}$ is the set of all $n\times n$ real matrices. It is well-known that (1.1) is equivalent to

$$\mathscr{H}\begin{bmatrix} I_n \\ X \end{bmatrix} := \begin{bmatrix} A & -G \\ -H & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix}(A-GX), \tag{1.2}$$

where $I_n$ is the $n\times n$ identity matrix. The equation (1.2) implies that for any solution $X$ to (1.1), the column space of $\begin{bmatrix} I_n \\ X \end{bmatrix}$ is an $n$-dimensional invariant subspace of $\mathscr{H}\in\mathbb{R}^{2n\times 2n}$ associated with its eigenvalues that are those of $A-GX$. More than that, (1.2) leads to

$$\mathscr{H}\begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix}\begin{bmatrix} A-GX & -G \\ 0 & -(A^{\mathrm{T}}-XG) \end{bmatrix},$$

implying the eigenvalues of $\mathscr{H}$ is the union of the eigenvalues of $A-GX$ and $-(A^{\mathrm{T}}-XG)$.

The matrix $\mathscr{H}$ in (1.2) happens to be real and Hamiltonian, i.e., satisfying

$$\mathscr{H}\mathscr{J}_n = -\mathscr{J}_n\mathscr{H}^{\mathrm{T}} \quad \text{with} \quad \mathscr{J}_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

As a real Hamiltonian matrix, its eigenvalues come in quadruples $(\lambda,\bar{\lambda},-\lambda,-\bar{\lambda})$, unless $\lambda$ is purely imaginary in which case, it comes in pairs $(\lambda,-\lambda)$. Under certain conditions from the $\mathcal{H}_2$-optimal control, indeed $\mathscr{H}$ has no eigenvalues on the imaginary axis, and then $\mathscr{H}$ has precisely $n$ eigenvalues in $\mathbb{C}_-$ (the open left half-plane) and $n$ eigenvalues in $\mathbb{C}_+$ (the open right half-plane). The solution to (1.1) of interest is the one for which the eigenvalues of $A-GX$ consist of exactly those of $\mathscr{H}$ in $\mathbb{C}_-$. Denote by $\Phi$ this special solution. It can be shown [12] that $\Phi^{\mathrm{T}}=\Phi$, and the eigenvalues of $-(A^{\mathrm{T}}-\Phi G) = -(A-G\Phi)^{\mathrm{T}}$ are precisely the opposites of the ones of $A-G\Phi$.

The doubling algorithms, originally proposed in 1970s for solving CARE and others (see the short survey [3]) but elegantly reformulated in [4] in 2005, turn out to be very much the methods of choice these days to compute the solution $\Phi$ for $n$ up to a couple of thousands. Chu, Fan, and Lin [4] also named their reformulation as *structure-preserving doubling algorithm* (SDA) to reflect its structure-preserving feature. For fast convergence of SDA, in the preset-up we have to find an 1-parameter-dependent linear fractional transformation

$$z\in\mathbb{C}\to w(z;\gamma) = \frac{z-\gamma}{z+\gamma} \tag{1.3}$$

that can bring all eigenvalues of $\mathscr{H}$ lying in $\mathbb{C}_-$, i.e., those in $\operatorname{eig}(A-G\Phi)$, the multiset of the eigenvalues of $A-G\Phi$, into the interior of the unit disk, and the asymptotic convergence speed of the doubling iteration is measured by the maximal distance from the