On Finite Groups Whose Nilpotentisers Are Nilpotent Subgroups

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Abstract. Let \( G \) be a finite group and \( x \in G \). The nilpotentiser of \( x \) in \( G \) is defined to be the subset \( \text{Nil}_G(x) = \{ y \in G : [x, y] \text{ is nilpotent} \} \). \( G \) is called an \( N \)-group (\( n \)-group) if \( \text{Nil}_G(x) \) is a subgroup (nilpotent subgroup) of \( G \) for all \( x \in G \setminus Z^*(G) \) where \( Z^*(G) \) is the hypercenter of \( G \). In the present paper, we determine finite \( N \)-groups in which the centraliser of each noncentral element is abelian. Also we classify all finite \( n \)-groups.

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1 Introduction

Consider \( x \in G \). The centraliser, nilpotentiser and engeliser of \( x \) in \( G \) are

\[
C_G(x) = \{ y \in G : [x, y] \text{ is abelian} \}, \quad \text{Nil}_G(x) = \{ y \in G : [x, y] \text{ is nilpotent} \}
\]

and

\[
E_G(x) = \{ y \in G : [y, n x] = 1 \text{ for some } n \}
\]

respectively. Obviously

\[
C_G(x) \subseteq \text{Nil}_G(x) \subseteq E_G(x) \quad \text{for each } x \in G.
\]

Note that \( \text{Nil}_G(x) \) and \( E_G(x) \) are not necessarily subgroups of \( G \). So determining the structure of groups by nilpotentisers (or engelisers) is more complicated than the centralisers. Let \( G \) be a finite group. Let \( 1 \leq Z_1(G) < Z_2(G) < \cdots \) be a series of subgroups of \( G \), where \( Z_1(G) = Z(G) \) is the center of \( G \) and \( Z_{i+1}(G) \), for \( i > 1 \), is defined by

\[
\frac{Z_{i+1}(G)}{Z_i(G)} = Z\left( \frac{G}{Z_i(G)} \right).
\]

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Let $Z^*(G) = \bigcup_i Z_i(G)$. The subgroup $Z^*(G)$ is called the hypercenter of $G$. We say a group is $n$-group in which $\text{Nil}_G(x)$ is a nilpotent subgroup for each $x \in G \setminus Z^*(G)$.

Now a group is $N$-group in which the nilpotentiser of each element is subgroup and a $CA$-group is a group in which the centraliser of each noncentral element is abelian (see [16] or [5]). The class of $N$-groups were defined and investigated by Abdollahi and Zarrin in [1]. In particular they showed that every centerless $CA$-group is an $N$-group.

In this paper, we shall prove the following generalisation of this result.

**Theorem 1.1.** Let $G$ be a nonabelian $CA$-group. Then $G$ is an $N$-group if and only if we have one of the following types:

1. $G$ has an abelian normal subgroup $K$ of prime index.
2. $\frac{G}{Z(G)}$ is a Frobenius group with Frobenius kernel $K$ and Frobenius complement $L$, where $K$ and $L$ are abelian.
3. $\frac{G}{Z(G)}$ is a Frobenius group with Frobenius kernel $K$ and Frobenius complement $L$, such that $K = PZ$, where $P$ is a normal Sylow $p$-subgroup of $G$ for some prime divisor $p$ of $|G|$, $P$ is a $CA$-group, $Z(P) = P \cap Z$ and $L = HZ$, where $H$ is an abelian $p'$-subgroup of $G$.
4. $\frac{G}{Z(G)} \cong \text{PSL}(2,q)$ and $G' \cong \text{SL}(2,q)$ where $q > 3$ is a prime-power number and $16 \nmid q^2 - 1$.
5. $\frac{G}{Z(G)} \cong \text{PGL}(2,q)$ and $G' \cong \text{SL}(2,q)$ where $q > 3$ is a prime and $8 \nmid q \pm 3$.
6. $G = P \times A$ where $A$ is abelian and $P$ is a nonabelian $CA$-group of prime-power order.

A group is said to be an $E$-group whenever engeliser of each element of such group is subgroup. The class of $E$-groups was defined and investigated by Peng in [13,14]. Also Heineken and Casolo gave many more results about them (see [3,4,6]). Now recall that an engel group is a group in which the engeliser of every elements is the whole group. If $G$ is an $E$-group such that the engeliser of every element is engel, $G$ is an $n$-group since every finite engel group is nilpotent. This result motivates us to classify all finite $n$-groups in following theorem.

But before giving it, recall that the Hughes subgroup of a group $G$ is defined to be the subgroup generated by all the elements of $G$ whose orders are not $p$ and denoted by $H_p(G)$ where $p$ is a prime. Also a group $G$ is said to be of Hughes-Thompson type, if for some prime $p$ it is not a $p$-group and $H_p(G) \neq G$.

**Theorem 1.2.** Let $G$ be a nonnilpotent group. Then $G$ is an $n$-group if and only if $\frac{G}{Z^*(G)}$ satisfies one of the following conditions:

1. $\frac{G}{Z^*(G)}$ is a group of Hughes-Thompson type and

$$\left| \text{Nil}_{\frac{G}{Z^*(G)}}(xZ^*(G)) \right| = p$$

for all $xZ^*(G) \in \frac{G}{Z^*(G)} \setminus H_p(\frac{G}{Z^*(G)})$;