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The Distributional Henstock-Kurzweil Integral and Applications: a Survey

Guoju Ye and Wei Liu*

College of Science, Hohai University, Nanjing 210098, P. R. China

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Abstract. This study presents a summary of the current state of research on the distributional Henstock-Kurzweil integral. Basic properties such as integration by parts, Hake theorem, inner product, Hölder inequality, second mean value theorem, orderings, Banach lattice, convergence theorems, fixed point theorems, are shown. This study also summarizes its applications in integral and differential equations.

AMS subject classifications: 45G10, 26A42, 46F30, 46F05

Key words: Distributional Henstock-Kurzweil integral, Banach lattice, convergence theorem, cone, fixed point theorem, integral and differential equations.

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1 Introduction

In the integration theory, there is a simple way to define integrals, defined by their primitives. Take the Lebesgue integral as an example, if a function f is the derivative of an absolute continuous (AC) function F, then f is Lebesgue integrable. In symbol, if $F \in AC$ and $F' = f$ a.e., then $f \in L$. The Henstock-Kurzweil (HK) integral has a similar definition, that is, if $F \in ACG^*$ and $F' = f$ a.e., then $f \in HK$ ([1–7]). However, if the primitive F is a continuous function or a regulated function, i.e., $F \in C$ or $F \in G$ (here C and G denote the spaces of the continuous functions and regulated functions, respectively), then the Schwartz distribution (or generalized function) and distributional derivative are needed here ([9, 10, 13, 15, 20]), because there are plenty of continuous functions that are differentiable nowhere ([32]). For simplicity, relationships between primitives and integrands for some major integrals are shown in Figure 1.

This survey is an outline of some results on the distributional Henstock-Kurzweil integral (for short, D_{HK}). Namely, a distribution f is distributionally Henstock-Kurzweil integrable on an interval $[a,b]$ if there exists a continuous function F such that the distributional derivative of F is f and denote by $\int_a^b f = F(b) - F(a)$, and F is called the

*Corresponding author. *Email addresses:* `yegj@hhu.edu.cn` (G. Ye), `liuw626@hhu.edu.cn` (W. Liu)

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\begin{array}{lccccccccc}
\textit{Primitive } F: & C^1 & \subsetneq & AC & \subsetneq & ACG^* & \subsetneq & C & \subsetneq & G \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\textit{Integrand } f: & C & \subsetneq & L & \subsetneq & HK & \subsetneq & D_{HK} & \subsetneq & G_{HK}
\end{array}
$$

Figure 1: Relationships between primitives and integrands for some major integrals.

primitive of f. This integral seems to have been first introduced by Mikusiński and Ostaszewski [11]. Then, it was developed in detail in the plane by Ang, Schmitt and Vy [12] and on $[-\infty,+\infty]$ by Talvila [13].

According to its definition, this integral comprises Riemann, Lebesgue, Henstock-Kurzweil, Perron, Denjoy, and improper integrals as special cases ([1–7, 12, 13, 20]). Denote the spaces of the Henstock-Kurzweil integrable functions and the Henstock-Kurzweil integrable distributions by HK and D_{HK}, respectively. The space D_{HK} is a separable Banach space with the Alexiewicz norm and it is isometrically isomorphic to the space of continuous functions on a closed interval with uniform norm. The spaces L and HK are dense in the space D_{HK}. Some other basic properties such as integration by parts, Hake theorem, inner product, Hölder inequality, second mean value theorem are also introduced in Section 2. Moreover, orderings and Banach lattice are shown in Section 3. Section 4 is devoted to the weak and strong convergence theorems and quasi-convergence. In the last section, we show that D_{HK} is an ordered Banach space with a regular cone under certain ordering and then give a fixed point theorem in D_{HK}. Applications in functional Urysohn integral equation, Darboux problem and Measure differential equation are presented.

2 Basic definitions and preliminaries

For convenience, we use the same notations as in [20] and list some basic facts as follows.

Let (a,b) be an open interval in $\mathbb{R}$, we define

$$\mathcal{D}((a,b))=\Big\{\phi:(a,b)\to\mathbb{R} \mid \phi\in C_c^\infty \text{ and } \phi \text{ has a compact support in } (a,b)\Big\}.$$

The distributions on (a,b) are defined to be the continuous linear functionals on $\mathcal{D}((a,b))$. The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

For all $f\in\mathcal{D}'((a,b))$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f',\phi\rangle=-\langle f,\phi'\rangle$, where $\phi\in\mathcal{D}((a,b))$ is a test function. Further, we write distributional derivative as f' and its pointwise derivative as $f'(t)$ where $t\in\mathbb{R}$. From now on, all derivative in this paper will be distributional derivatives unless stated otherwise.

Denote the space of continuous functions on $[a,b]$ by $C([a,b])$. Let

$$C_0=\Big\{F\in C([a,b]):F(a)=0\Big\}. \tag{2.1}$$