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Metric Subregularity for a Multifunction

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Abstract. Metric subregularity is an important and active area in modern variational analysis and nonsmooth optimization. Many existing results on the metric suregularity were established in terms of coderivatives of the multifunctions concerned. This note tries to give a survey of the metric subregularity theory related to the coderivatives and normal cones.

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1 Introduction

Let X and Y be Banach spaces and $\Phi: X \rightrightarrows Y$ be a multifunction such that its graph $\mathrm{gph}(\Phi) := \{(x,y) \in X \times Y : y \in \Phi(x)\}$ is closed. Recall that Φ is metrically subregular at $(a,b) \in \mathrm{gph}(\Phi)$ if there exist $\tau, \delta \in (0,+\infty)$ such that

$$d(x,\Phi^{-1}(b)) \leq \tau d(b,\Phi(x)) \quad \forall x \in B(a,\delta), \tag{1.1}$$

where $d(x,\Phi^{-1}(b)) := \inf\{\|x-u\| : u \in \Phi^{-1}(b)\}$ and $B(a,\delta) := \{u \in X : \|x-a\| < \delta\}$. This property provides an estimate of how far a candidate x can be from the solution set of generalized equation (GE)

$$b \in \Phi(x). \tag{GE}$$

Also recall that a multifunction $M: Y \rightrightarrows X$ is said to be calm at $(b,a) \in \mathrm{gph}(M)$ if there exists $L \in (0,+\infty)$ such that

$$d(x,M(b)) \leq L\|y-b\| \quad \text{for all } (y,x) \in \mathrm{gph}(M) \text{ close to } (b,a).$$

It is known that Φ is metrically subregular at (a,b) if and only if $M = \Phi^{-1}$ is calm at (b,a) (cf. [9]). The metric subregularity and calmness have been already studied by many authors under various names (see [2, 3, 7–10, 14–19, 25, 26, 28–31] and therein references).

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Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and consider the special case that $Y = \mathbb{R}$, $b = 0$ and

$$\Phi(x) := [f(x), +\infty) \quad \forall x \in X. \tag{1.2}$$

In this case, generalized equation (GE) reduces to the following inequality

$$\text{(IE)} \qquad f(x) \leq 0,$$

while metric subregularity (1.1) reduces to

$$d(x,S) \leq \tau [f(x)]_+ \quad \forall x \in B(a,\delta), \tag{1.3}$$

where $S = \{x \in X | f(x) \leq 0\}$ and $[f(x)]_+ = \max\{f(x), 0\}$. Usually inequality (IE) is said to have a local error bound at a if there exist $\tau, \delta \in (0, +\infty)$ such that (1.3) holds. Error bound properties have important applications in sensitivity analysis and convergence analysis of mathematical programming. The research on error bounds has attracted the interest of many researchers and there are a vast number of publications reporting the progress in this area (cf. [4, 11, 12, 20–22, 24, 26, 32,35] and references therein). In particular, studies on error bounds have been well carried out in terms of subdifferentials; these studies are mainly carried out in two directions of approach. The first direction is described by the subdifferentials of f at points *inside* the solution set S and the normal cones of S. In this direction, it is known that if f is convex then inequality (IE) has a local error bound at a if and only if there exist $\tau, \delta \in (0, +\infty)$ such that

$$N(S,x) \cap B_{X^*} \subset [0, \tau] \partial f(x) \quad \forall x \in S \cap B(a,\delta)$$

(cf. [5, 11, 12, 20, 28]). The second direction is described only by the subdifferentials of f at points *outside* the solution set S. In this direction, Ioffe [13] first studied error bound (under a different name) and proved that the following implication holds:

$$d(0, \partial_c f(x)) \geq \kappa \quad \forall x \in B(a,\delta) \setminus S \Longrightarrow \text{(IE) has a local error bound at } a. \tag{1.4}$$

Note that the coderivative for a multifunction is the counterpart of the subdifferential for a real-valued function and that the subdifferential $\partial f(x)$ of f at x is equal to the coderivative $D^*\Phi(x, f(x))(1)$ (where Φ is defined by (1.2)). So it is natural to study the more general metric subregularity for a closed multifunction between two Banach spaces in terms of coderivatives also along two directions of approach. In this note, we will give a survey of the research of the metric subregularity along these two directions.

2 Preliminaries

Let X be a Banach space with topological dual X^*. Let B_X and S_X denote the closed unit ball and unit sphere of X, respectively. For a closed subset A of X and a point x in A,