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Legendre Pseudospectral Approximation of Boussinesq Systems and Applications to Wave Breaking

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Abstract. In this paper, we propose a spectral projection of a regularized Boussinesq system for wave propagation on the surface of a fluid. The spectral method is based on the use of Legendre polynomials, and is able to handle time-dependent Dirichlet boundary conditions with spectral accuracy.
The algorithm is applied to the study of undular bores, and in particular to the onset of wave breaking connected with undular bores. As proposed in [2], an improved version of the breaking criterion recently introduced in [5] is used. This tightened breaking criterion together with a careful choice of the relaxation parameter yields rather accurate predictions of the onset of breaking in the leading wave of an undular bore.

AMS subject classifications: 35Q53, 65M70, 76B15, 76B25

Key words: Boussinesq system, Legendre projection, undular bore, wave breaking, boundary conditions, spectral accuracy.

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1 Introduction

In this paper, we report on the application of a long-wave system of evolution equations to the study of weak undular bores. We propose a spectral projection of a regularized Boussinesq system which is a model for wave propagation on the surface of a fluid. The numerical method is put to use for the simulation of undular bores, and the prediction of the onset of breaking of the leading wave at the bore front. The current work is an

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improvement over an earlier study by two of the authors [5], in which the onset of wave breaking was studied using a finite-difference method. The spectral method used in the present paper is far more accurate than the finite difference-method used in [5], and a fine-tuned breaking criterion, such as proposed in [2] leads to a more accurate prediction of incipient breaking of the leading wave.

Boussinesq systems are widely used in the modeling of surface water waves, especially in the context of nearshore wave prediction and coastal modeling. The validity of Boussinesq systems as models for water waves depend on two nondimensional parameters quantifying the predominant waveheight and wavelength of a wavefield under study. If the undisturbed depth is given by h_0, a typical amplitude is a, and a typical wavelength is λ, then we define the amplitude parameter $\alpha = a/h_0$, and the shallowness parameter $\beta = (h_0/\lambda)^2$. In the Boussinesq scaling regime, introduced in [11], it is assumed that both α and β are small and of the same order of magnitude. However, for many wave fields occurring in practice, these parameters may not always be small, and significant efforts have been made to extend the range of applicability, both in the direction of shorter waves (β not small), and in the direction of larger amplitudes (α not small) [18, 22, 24, 26, 29, 33]. Generally, Boussinesq models incorporate two unknowns: the deflection of the free surface, and one other dependent variable. In the present work, we are interested in a Boussinesq system formulated in surface elevation - velocity variables, which is easily solved and known to be well posed ([8, 14]) even in the presence of boundary conditions. The dispersive system under study is

$$\begin{aligned} &\eta_t + h_0 u^\theta_x + (\eta u^\theta)_x - \frac{h_0^2}{2}\left(\theta^2 - \frac{1}{3}\right)\eta_{xxt} = 0, \\ &u^\theta_t + g\eta_x + u^\theta u^\theta_x - \frac{h_0^2}{2}(1-\theta^2)u^\theta_{xxt} = 0. \end{aligned} \tag{1.1}$$

In this system, $u^\theta(x,t)$ represents the horizontal component of the fluid velocity at a height $0 < \theta h_0 < h_0$ in the fluid column, and $\eta(x,t)$ describes the displacement of the free surface from its rest position. As mentioned above, h_0 is the undisturbed depth of water. Finally, g denotes the gravitational acceleration as usual. The system (1.1) represents one particular class of the models put forward in [7], and has been chosen for the present study because it appears most convenient from a numerical point of view. In [15, 25], the system was also extended to three dimensional flows, and moving bottom profiles.

In the present work, the main application of the Legendre projection of (1.1) is the study of an undular bore. Quite generally, a bore is a transition between two uniform free-surface flows of different flow depths. If the difference in flow depths is a_0, and assuming that one of these depths is at the undisturbed water level h_0, the incident water level is given by $h_0 + a_0$, and the strength of the bore is given by $\alpha_0 = a_0/h_0$. In the current context, we consider bores propagating into previously undisturbed water in a narrow channel of uniform depth and width. This geometric setup is called for in order to ensure that the flow is not affected by bottom impurities and transverse effects.

In an experimental and theoretical study, it was found in [16] that depending on the value of α_0, bores propagating into undisturbed water fall into three categories. When α_0