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On Unitary Invariant Strongly Pseudoconvex Complex Landsberg Metrics

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Abstract. In this paper, we prove that a unitary invariant strongly pseudoconvex complex Finsler metric is a complex Landsberg metric if and only if it comes from a unitary invariant Hermitian metric. This implies that there does not exist unitary invariant complex Landsberg metric unless it comes from a unitary invariant Hermitian metric.

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Key words: Unitary invariant, complex Landsberg metric, complex Finsler metric.

1 Introduction

In real Finsler geometry, a Berwald metric is necessary a Landsberg metric. It is still an open problem that whether there exists a Landsberg metric which does not come from a Berwald metric [2]. This problem is also called the unicorn problem by D. Bao [3] and M. Matsumoto [2].

M. Matsumoto conjectured that there does not exist unicorn metric, which implies that every Landsberg metric comes from a Berwald metric. In 2008, Z. I. Szabó claimed that all regular Landsberg metrics are Berwald metrics [4]. A gap, however, was soon found in the proof by himself [5], thus leaving the problem still open.

On the other hand, in [6,7], G. S. Asnov constructed a family of almost regular unicorn metrics which come from (α,β)-metrics. In 2009, Z. Shen [8] characterized almost regular Landsberg (α,β)-metrics which generalized G. S. Asanovs results. For the spherically symmetric real Finsler metrics which are not necessary (α,β)-metrics, X.-H. Mo and L.-F. Zhou [9] proved that there does not exist any non-Berwaldian Landsberg metrics among the regular case.

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In complex Finsler geometry, there are also notions of complex Berwald metric, weakly complex Berwald metric, and complex Landsberg metric. It was known that every Kähler-Berwald metric is necessary a complex Landsberg metric [10]. One may wonder whether there exists a complex Landsberg metric which does not come from a Kähler-Berwald metric?

Unlike in real Finsler geometry, there are few explicit examples of strongly pseudoconvex complex Finsler metrics in literatures. This situation has already been changed because of the recent work of C.-P. Zhong [12], where unitary invariant strongly pseduoconvex complex Finsler metrics were systematically studied and the explicit method of constructing strongly psuedoconvex or even strongly convex complex Finsler metrics were given. In [13], H.C. Xia and C.-P. Zhong gave a classification of unitary invariant weakly complex Berwald metrics which are of constant holomorphic curvatures. It was proved in [12] that there is neither complex Berwald metric nor Kähler Finsler metric which is unitary invariant and does not come from a Hermitian metric. There are, however, lots of weakly complex Berwald metrics which are unitary invariant and they do not come from Hermitian metrics. One may wonder whether there exists unitary invariant complex Landsberg metric which does not come from a Kähler-Berwald metric.

In this paper, we prove that a unitary invariant strongly pseudoconvex complex Finsler metric is a complex Landsberg metric if and only if it comes from a unitary invariant Hermitian metric. This implies that there does not exist unitary invariant complex Landsberg metric unless it comes from a unitary invariant Hermitian metric.

2 Preliminary

Let $\mathbb{C}^n$ be a complex n dimensional linear space, denote by $\langle\cdot,\cdot\rangle$ the canonical complex Euclidean inner product and $\|\cdot\|$ the induced norm in $\mathbb{C}^n$. Let F be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $D\subset\mathbb{C}^n$. It was proved in [12] that F is unitary invariant if and only if there exists a smooth function $\phi(t,s):[0,+\infty]\times[0,+\infty]\to(0,+\infty)$ such that $F=\sqrt{r\phi(t,s)}$ with

$$r=\|v\|^2,\ t=\|z\|^2,\ s=\frac{|\langle z,v\rangle|^2}{\|v\|^2},$$

where $z=(z^1,\cdots,z^n)\in D$ and $v=(v^1,\cdots,v^n)\in T_z^{1,0}D$.

Lemma 2.1. *[12] Let $F=\sqrt{r\phi(t,s)}$ be a strongly pseudoconvex complex Finsler metrics defined on a domain $D\subset\mathbb{C}^n$. Then the fundamental tensor of F is*

$$G_{\gamma\overline{\tau}}=(\phi-s\phi_s)\delta_{\gamma\overline{\tau}}+r\phi_{ss}s_\gamma s_{\overline{\tau}}+\phi_s\overline{z^\gamma}z^\tau. \tag{2.1}$$

It is known that for a strongly pseudoconvex complex Finsler metric F, there are several complex Finsler connections associated to it. The most often used complex Finsler connections are the Chern-Finsler connection [1], the complex Rund connection and the