Composite Implicit Iteration Process for Asymptotically Hemi-Pseudocontractive Mappings

Ling Luo and Weiping Guo *

School of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou 215009, Jiangsu Province, P. R. China.

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Abstract. In Banach space, the composite implicit iterative process for uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mappings are studied, and the sufficient and necessary conditions of strong convergence for the composite implicit iterative process are obtained.

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1 Introduction and preliminaries

Throughout this work, we assume that $E$ is a real Banach space. $E^*$ is the dual space of $E$ and $J:E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^*: <x,f> = \|x\| \|f\|, \|f\| = \|x\| \}, \quad \forall x \in E,$$

where $<\cdot,\cdot>$ denotes duality pairing between $E$ and $E^*$. A single-valued normalized duality mapping is denoted by $j$.

Let $C$ be a nonempty subset of $E$ and $T:C \to C$ a mapping, we denote the set of fixed points of $T$ by $F(T) = \{ x \in C; Tx = x \}$.

Definition 1.1. ([1]) $T$ is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C \text{ and } n \geq 1.$$

*Corresponding author. Email addresses: guoweiping18@aliyun.com (W. Guo), luoling19901120@163.com (L. Luo)
(2) ([2]) \( T \) is said to be uniformly \( L \)-Lipschitzian, if there exists \( L > 0 \) such that
\[
\|T^nx - T^ny\| \leq L\|x - y\|, \quad \forall x, y \in C \text{ and } n \geq 1.
\]

(3) ([3]) \( T \) is said to be asymptotically pseudocontractive, if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \), for any \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2, \quad n \geq 1.
\]

(4) ([4]) \( T \) is said to be asymptotically hemi-pseudocontractive, if \( F(T) \neq \emptyset \) and there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that, for any \( x \in C \) and \( p \in F(T) \), there exists \( j(x - p) \in J(x - p) \) such that
\[
\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2, \quad n \geq 1.
\]

**Remark 1.1.** It is easy to see that if \( T \) is an asymptotically nonexpansive mapping, then \( T \) is a uniformly \( L \)-Lipschitzian and asymptotically pseudocontractive mapping, where \( L = \sup_{n \geq 1} \{k_n\} \); if \( T \) is an asymptotically pseudocontractive mapping with \( F(T) \neq \emptyset \), then \( T \) is an asymptotically hemi-pseudocontractive mapping.

Let \( C \) be a nonempty closed convex subset of \( E \) and \( T : C \to C \) be a uniformly \( L \)-Lipschitzian asymptotically hemi-pseudocontractive mapping, for any given \( x_1 \in C \), we introduce a composite implicit iteration process \( \{x_n\} \) as follows:
\[
\begin{cases}
  x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^nx_n, \\
  y_n = (1 - \beta_n)x_n + \beta_n T^nx_{n+1},
\end{cases}
\quad \forall n \geq 1,
\tag{1.1}
\]
where \( \{\alpha_n\}, \{\beta_n\} \) are two real sequences in \([0,1]\).

As \( \beta_n = 0 \) for all \( n \geq 1 \), then (1.1) reduces to
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^nx_n.
\tag{1.2}
\]

**Remark 1.2.** For any given \( x_1 \in C \), define the mapping \( A_n : C \to C \), such as:
\[
A_n x = (1 - \alpha_n)x_n + \alpha_n T^n[(1 - \beta_n)x_n + \beta_n T^n x], \quad \forall x \in C,
\]
where \( C \) is a nonempty closed convex subset of \( E \) and \( T : C \to C \) is a uniformly \( L \)-Lipschitzian. Then
\[
\|A_n x - A_n y\| = \|\alpha_n(T^n[(1 - \beta_n)x_n + \beta_n T^n x] - T^n[(1 - \beta_n)x_n + \beta_n T^n y])\| \\
\leq \alpha_n \beta_n L \|T^n x - T^n y\| \\
\leq \alpha_n \beta_n L^2 \|x - y\|
\]
for all \( x, y \in C \). Thus \( A_n \) is a contraction mapping if \( \alpha_n \beta_n L^2 < 1 \) for all \( n \geq 1 \), and so there exists a unique fixed point \( x_{n+1} \in C \) of \( A_n \), such that \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n[(1 - \beta_n)x_n + \beta_n T^n x_{n+1}] \). This shows that the composite implicit iteration process (1.1) is well defined.