J. Math. Study
doi: 10.4208/jms.v48n4.15.02

Vol. **48**, No. 4, pp. 330-344
December 2015

Cauchy Matrices in the Observation of Diffusion Equations

Faker Ben Belgacem[1,*] and Sidi-Mahmoud Kaber[2,*]

[1] *Laboratoire de Mathématique Appliquée de Compiègne, Sorbonne Universités, Compiègne F-60205, UTC, EA 2222, France.*
[2] *Laboratoire Jacques-Louis Lions, Sorbonne Universités, Paris F-75005, UPMC Univ Paris 06, UMR 7598, France.*

Received 21 July, 2015; Accepted 22 September, 2015

Abstract. Observability Gramians of diffusion equations have been recently connected to infinite Pick and Cauchy matrices. In fact, inverse or observability inequalities can be obtained after estimating the extreme eigenvalues of these structured matrices, with respect to the diffusion semi-group matrix. The purpose is hence to conduct a spectral study of a subclass of symmetric Cauchy matrices and present an algebraic way to show the desired observability results. We revisit observability inequalities for three different observation problems of the diffusion equation and show how they can be (re)stated through simple proofs.

AMS subject classifications: 15B05, 15B34, 93B07, 35K05

Key words: Pick matrices, Cauchy matrices, Hadamard product, diffusion equation, observability inequalities.

1 Introduction

Observability inequalities are the milestone for the HUM method applied to null-controll -ability problems (see [20,21]). These inequalities are hard to prove in particular for infinite dimensional parabolic problems. A lot of work have been done for the observability of the heat equation. Consult the non-exhaustive list [8,10–12,17–19,24,26,27]. We are, of course, far from supplying a state-of-the-art on the subject.

We revisit some of the observability estimates in the diffusion problems in one dimension and give alternative proofs using algebraic tools such as spectral estimates of some structured matrices. Connection between the controllability of parabolic problems and Cauchy and Pick matrices has been pointed out in [1]. This remark allowed to state the

*Corresponding author. *Email address:* `faker.ben-belgacem@utc.fr` (F. Ben Belgacem), `kaber@ljll.math.upmc.fr` (S. M. Kaber)

severe ill-posedness degree of the exact controllability of the heat equation. A deeper analysis of these matrices results in algebraic proofs of the observability inequalities. The central element in the analysis is the closed form of the inverse of Cauchy matrices (see [6]). The first observation problem we study is the one analyzed in [8]. The second one is a boundary observation problem investigated in [1, 24]. The third and last one is the distributed observability (see, eg, [36]). Notice that for this last problem, the spectral analysis of the structured matrices needs to be complemented by Schur's theorem to provide bounds of the eigenvalues of the Hadamard entrywise product of matrices.

An outline of the paper is as follows. In Section 2, we investigate spectral properties of a countably infinite Pick matrix with respect to the diffusion semi-group matrix. The spectral equivalence between Pick matrix and symmetric Cauchy matrix is stated in the preliminary step. Then, we take profit of the closed form of the inverse of Cauchy matrices to state the desired comparison result. In Section 3, three observation problems are revisited. We follow the methodology by Fattorini and Russel [10]. However, instead of considering a moment equation as in [8, 12, 34, 36], we rather take advantage of the connection between the observability inequality for each problem and the spectral properties of structured matrices. Alternative proofs are hence exposed owing to algebraic estimates developed in the previous section. As a result, we bring about easy proofs to the observation of the heat problem.

Notation 1.1. Let $\ell^2(\mathbb{R})$ be the Hilbert space of countably infinite real sequence $\boldsymbol{\psi} = (\psi_k)_{k\geq 1}$ that are square summable and denote its norm by $\|\cdot\|_{\ell^2}$. For a given integer $N>1$, we will use the symbol $\ell^2_N(\mathbb{R})$ for the sub-space in $\ell^2(\mathbb{R})$ involving the sequence $\boldsymbol{\psi}$ whose entries $(\psi_k)_{k\geq N+1}$ vanish. It is isomorphic to the standard space $\mathbb{R}^N$ endowed with the Euclidean norm. The closure of the union of the subspaces $\ell^2_N(\mathbb{R})$ is dense in the whole space $\ell^2(\mathbb{R})$. Next, we consider a countably infinite matrix $\mathcal{C}$, defined by the real entries $(c_{k,m})_{k,m\geq 1}$. Throughout, the notation $\mathcal{C}_N^{(T)}$ stands for the principle sub-matrix with order N of $\mathcal{C}$. It may be identified to the infinite matrix

$$\mathcal{C}_N^{(T)} = \begin{pmatrix} (c_{k,m})_{1\leq k,m\leq N} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1}$$

If $\mathcal{P}_N$ stands for the canonical projection operator from $\ell^2(\mathbb{R})$ into $\ell^2_N(\mathbb{R})$ (equivalently on $\mathbb{R}^N$), we may write that $\mathcal{C}_N^{(T)} = \mathcal{P}_N \mathcal{C} \mathcal{P}_N$. Subsequently, we shall accept, in some places, a notation abuse concerning the principal sub-matrices of $\mathcal{C}_N^{(T)}$ with dimension N. It has either the form of (1.1), an infinite matrix representing an operator defined on $\ell^2_N(\mathbb{R})$, or simply the square matrix $(c_{k,m})_{1\leq k,m\leq N}$ with dimension N related to a linear application in $\mathbb{R}^N$. We refer to [9] for the fundamental properties of these spaces. Now, for any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L^2(0,T;X)$ the